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Another Class of Admissible Perturbations of Special Expressions

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Abstract

In this paper, another class of admissible perturbations of special expression $\mathbf{M}_0 = \sum_{\mathrm{k}=0}^{\mathrm{r}} \mathrm{c_k t^{\alpha_k} D_t^{\rho_k}}$ in the weighted space $\mathcal{L}^2_\omega([1,\infty))$ is being presented. This perturbation is denoted by $\mathbf{M}_1$ and is different from what has been presented in [3] and the one published [2]. In this work, it will be shown that the operator $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in the non-weighted space $\mathcal{L}^2([1,\infty))$, and eventually preserves the essential spectrum of $\mathbf{M}_0$ in that space. The discussion is limited to special expressions with $\alpha_1 < \rho_1$.

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1 Introduction

Consider the differential expressions in $\mathcal{L}^2[1,\infty)$ of the form

$$\mathbf{M}_0 = \sum_{\mathrm{k}=0}^{\mathrm{r}} \mathrm{c_k t^{\alpha_k} D_t^{\rho_k}} \tag{1}$$

where $c_k \in \mathbb{C}$ and $D_t = \frac{d}{dt}$ with
i) $\rho_k \in \mathbb{N}$ for every k, such that

$$0 = \rho_0 < \rho_1 < \cdots < \rho_r = n \tag{2}$$

ii) $\alpha_k \in \mathbb{R}$ for every k, satisfying

$$\alpha_0 = 0 \text{ and } \alpha_1 \leq \rho_1, \tag{3}$$

and

$$1 \geq \frac{\alpha_k - \alpha_{k-1}}{\rho_k - \rho_{k-1}} \geq \frac{\alpha_{k+1} - \alpha_k}{\rho_{k+1} - \rho_k}, \tag{4}$$

for $k = 1, ..., r-1$ if $r > 1$. If we denote by $\sigma_1 < \sigma_2 < \cdots < \sigma_{s-1}$ those indices $k(k = 1, 2, ..., r-1)$ for which the strict inequality holds in (4) and with $\sigma_0 = 0$ and $\sigma_s = r$, we have

$$\frac{\alpha_{\sigma_j} - \alpha_{\sigma_{j-1}}}{\rho_{\sigma_j} - \rho_{\sigma_{j-1}}} = \frac{\alpha_k - \alpha_{k-1}}{\rho_k - \rho_{k-1}}, \tag{5}$$

for $\sigma_{j-1} \leq k \leq \sigma_j$, $j = 1, ..., s$ if $s \geq 1$, and

$$\frac{\alpha_{\sigma_j} - \alpha_{\sigma_{j-1}}}{\rho_{\sigma_j} - \rho_{\sigma_{j-1}}} > \frac{\alpha_{\sigma_{j+1}} - \alpha_{\sigma_j}}{\rho_{\sigma_{j+1}} - \rho_{\sigma_j}}, \tag{6}$$

for $j = 1, ..., s-1$ if $s \geq 2$. The special expression $\mathbf{M}_0$, its essential part, and the polygonal path it generates are defined as follows (cf. [5]).

Definition 1.1. *A* **special expression** $\mathbf{M}_0$ *is a differential expression of the form* (1) *where* ρ_k *and* α_k *satisfy* (2), (3) *and* (4) *and the coefficients* $c_k \in \mathbb{C}$ *satisfy the properties that for* $k = 0, \ldots, \sigma_1 - 1$, c_k *may be an arbitrary complex constant and for* $k = \sigma_1, \ldots, r$,

$$c_k \in \mathbb{C} \setminus \{0\},$$

and

$$\sum_{\substack{\rho_\delta + \rho_\lambda = 2\sigma \\ \sigma_i \leq \delta, \lambda \leq \sigma_{i+1}}} (-1)^{\rho_\delta + \sigma} c_\delta c_\lambda \geq 0$$

where $\sigma = \rho_{\sigma_i}, \ldots, \rho_{\sigma_{i+1}}$, $i = 1, \ldots, s-1$. *The indices* $\sigma_1, \ldots, \sigma_s$ *are called* **kink indices**. *The* **essential part** *of* $\mathbf{M}_0$ *is given by*

$$\mathbf{M}_{0,0} = \sum_{\mathrm{k}=0}^{\sigma_1} \mathrm{c_k t^{\alpha_k} D_t^{\rho_k}}. \tag{7}$$

The special expression $\mathbf{M}_0$ can be visualized graphically. Plot the points (ρ_k, α_k) and $(\rho_{k+1}, \alpha_{k+1})$ in $\mathbb{R}^2$, where $k = 0, 1, \ldots, r$. If we connect the points (ρ_k, α_k) and $(\rho_{k+1}, \alpha_{k+1})$ by a line, the resulting graph is called the ***polygonal path generated by*** $\mathbf{M}_0$. If the slope of the line connecting the points $(\rho_{\sigma_{i-1}}, \alpha_{\sigma_{i-1}})$ and $(\rho_{\sigma_i}, \alpha_{\sigma_i})$ is denoted by m_{σ_i} $(i = 1, \ldots, s)$, then

$$1 \geq m_{\sigma_1} > m_{\sigma_2} > \cdots > m_{\sigma_s}.$$

Hence the polygonal path generated by $\mathbf{M}_0$ lies on or below the bisectrix. Furthermore, the polygonal path generated by $\mathbf{M}_0$ can be associated by the function $\gamma : [0, n] \to \mathbb{R}$ defined by

$$\gamma(k) = \frac{1}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \left\{ (k - \rho_{\sigma_i})\alpha_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - k)\alpha_{\sigma_i} \right\} \tag{8}$$

for $k \in [\rho_{\sigma_i}, \rho_{\sigma_{i+1}}],\ i = 0, 1, 2, \ldots, s-1$.

Schultze[6] evaluated the essential spectrum of $\mathbf{M}_0$ and showed that the essential spectrum of the essential part of $\mathbf{M}_0$ and the essential spectrum of $\mathbf{M}_0$ are indeed equal. In short, he had the following result for $\alpha_1 < \rho_1$:

Theorem 1.2. *Let M_0 be a special expression. Then, for $\alpha_1 < \rho_1$,*

$$\sigma_e(\mathbf{M}_0) = \sigma_{\mathrm{e}}(\mathbf{M}_{0,0}) = \Big\{ \sum_{\mathrm{k}=0}^{\sigma_1} \mathrm{c_k z^{\rho_k}} \mid \mathrm{Re\ z} = 0 \Big\}. \tag{9}$$

Also, in the usual $\mathcal{L}^2$-space, a different class of perturbations, called *admissible perturbations* of special expressions $\mathbf{M}_0$ was determined by Mumpar-Victoria [4] in her paper, and was shown to preserve essential spectrum.

Definition 1.3. *Let* $\mathbf{M}$ *be a differential expression of the form*

$$M = \sum_{l=0}^{n-1} a_l(t) D_t^l. \tag{10}$$

We say that $\mathbf{M}$ *is an* **admissible perturbation** *of the special expression* $\mathbf{M}_0$ *if there exists a B such that the coefficients $a_l(t)$ satisfy the following*

$$\sup_{[x,x+1] \subset \mathrm{I}} \int_x^{x+1} \left| \frac{a_l(t)}{b_l(t)} \right|^2 dt < B \tag{11}$$

where $a_l(t) \in C^l(\mathrm{I})$ for $l = 0, ..., n-1$ and $0 < b_l(t)$ is an auxiliary function in $C^\infty(\mathrm{I})$ satisfying

$$b_l(t) = o(t^{\gamma(l+1)}) \text{ and } b_l(t) = o(t^{\gamma(l)}) \tag{12}$$

as $t \to \infty$.

Theorem 1.4. *Let* $\mathbf{M}_0$ *be a special expression and* $\mathbf{M}$ *an admissible perturbation of* $\mathbf{M}_0$ *of the form* (10) *satisfying* (11) *and* (12). *Then*

$$\sigma_e(\mathbf{M}_0 + \mathbf{M}) = \sigma_{\mathrm{e}}(\mathbf{M}_0).$$

2 The Weighted $\mathcal{L}^2$-Space

The results of Schultze were generalized in the weighted $\mathcal{L}^2$-space by Agapito [1]. In this section we recall this space and present some of Agapito's results.

Definition 2.1. *Suppose the function* $\omega : [1, \infty) \longrightarrow (0, \infty)$ *is measurable. The* **space** $\mathcal{L}^2_\omega(\mathrm{I})$ **of weighted square integrable functions** *over* $\mathrm{I} = [1, \infty)$ *is defined by*

$$\mathcal{L}^2_\omega(\mathrm{I}) := \left\{ \mathrm{f} : \mathrm{I} \longrightarrow \mathbb{C} \middle| \mathrm{f} \text{ is measurable and } \int_{\mathrm{I}} |\mathrm{f}(\mathrm{t})|^2 \, \omega(\mathrm{t})\mathrm{dt} < \infty \right\},$$

with inner product given by

$$\langle f, g \rangle_\omega := \int_I f(t)\overline{g(t)}\omega(t)dt$$

for any functions $f, g \in \mathcal{L}^2_\omega(\mathrm{I})$. $\mathcal{L}^2_\omega(\mathrm{I})$ *is commonly known as* **weighted** $\mathcal{L}^2$**-space** *with* ω *as the weight function.*

Remark 2.2. *It is easy to show that space* $\mathcal{L}^2_\omega(\mathrm{I})$ *together with the inner product* $< \cdot, \cdot >_\omega$ *is a Hilbert space. Furthermore, one can show that* $\mathcal{L}^2(\mathrm{I})$ *is equivalent to* $\mathcal{L}^2_\omega(\mathrm{I})$ *under the isometry* $W : \mathcal{L}^2_\omega(\mathrm{I}) \longrightarrow \mathcal{L}_2(\mathrm{I})$ *given by* $Wf = \omega^{\frac{1}{2}} f$.

In this work, we consider weight functions $\omega : [1, \infty) \longrightarrow (0, \infty)$ that satisfy the following conditions:
A_1) $\omega \in C^\infty([1, \infty))$.
A_2) $\frac{t^k \omega^{(k)}}{\omega} = \mathbf{O}(1)$ for $k = 1, 2, \ldots, n$.

A property of the weight function that is significant to our work is the following:

Lemma 2.3. *If* $\omega : [1, \infty) \longrightarrow (0, \infty)$ *satisfies* (A_1) *or* (A_2) *then so does* ω^α, *for any* $\alpha \in \mathbb{R}$.

3 Main Results

A new class of admissible perturbations of special expressions in the weighted space has been identified and it is of the form

$$\mathbf{M_1} = \sum_{\mathrm{l}=0}^{\mathrm{n}-1} \mathrm{a_l}(\mathrm{t})\mathrm{D}^{\mathrm{l}}_{\mathrm{t}} \qquad (13)$$

satisfying the following conditions:

AP$_1$) For every $i > l$,

$$\sup_{t\in \mathrm{I}} \left| \frac{a_i(t)}{t^{i-l} a_l(t)} \right| \quad exists$$

where $a_l(t) \in C^l(\mathrm{I})$, $\mathrm{I} = [1, \infty)$, $l = 0, 1, \dots, n-1$; and

AP$_2$) There exist auxiliary functions $b_l(t)$ such that

$$\left| \frac{a_l(t)}{b_l(t)} \right| \quad \text{is} \quad \text{bounded}$$

where $0 < b_l(t) \in C^\infty([1, \infty))$ for all l and $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$ as $t \longrightarrow \infty$.

It shall be presented in this work that if the weight function ω satisfies A_1 and A_2, and $\mathbf{M_1}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2_\omega(\mathrm{I})$ presented above, and that $\alpha_1 < \rho_1$, then $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ becomes an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2(\mathrm{I})$. Furthermore, $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ preserves the essential spectrum of the special expression $\mathbf{M}_0$ in $\mathcal{L}^2(\mathrm{I})$. This is the main result of the study.

Theorem 3.1. *Let* $\mathbf{M}_0$ *be a special expression of the form (1) with* $\alpha_1 < \rho_1$. *Let* $\mathbf{M_1}$ *having the form* (13) *that satisfies* (AP$_1$) *and* (AP$_2$) *be an admissible perturbation of* $\mathbf{M}_0$ *in* $\mathcal{L}^2_\omega([1, \infty))$. *If* ω *satisfies* A_1 *and* A_2, *then* $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ *is an admissible perturbation of* $\mathbf{M}_0$ *in* $\mathcal{L}^2([1, \infty))$. *In addition,* $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ *preserves the essential spectrum of* $\mathbf{M}_0$ *in* $\mathcal{L}^2([1, \infty))$.

Proof. Consider the admissible perturbation $\mathbf{M_1} = \sum_{\mathrm{l}=0}^{\mathrm{n}-1} \mathrm{a_l(t) D^l_t}$ of $\mathbf{M}_0$ presented above. By definition of that perturbation, $a_l(t) \in C^l(\mathrm{I})$, $l = 0, 1, \dots, n-1$. Also, this implies that $\left|\frac{a_l(t)}{b_l(t)}\right|$ is bounded. Hence, there exists an S such that $\left|\frac{a_l(t)}{b_l(t)}\right| < S$.

We first note that $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ is an operator that can be transformed as follows:

$$\begin{aligned}
\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}\mathrm{y} &= \omega^{\frac{1}{2}} \sum_{l=0}^{n-1} a_l(\omega^{-\frac{1}{2}} y)^{(l)} = \sum_{l=0}^{n-1}\sum_{i=0}^{l} \begin{pmatrix} l \\ i \end{pmatrix} a_l \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(l-i)} y^{(i)} \\
&= \sum_{l=0}^{n-1} \left(\sum_{l=i}^{n-1} \begin{pmatrix} l \\ i \end{pmatrix} a_l \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(l-i)} \right) y^{(i)} \\
&= \sum_{l=0}^{n-1} \left(\sum_{i=l}^{n-1} \begin{pmatrix} i \\ l \end{pmatrix} a_i \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)} \right) y^{(l)} \\
&= \sum_{l=0}^{n-1} (\mathbf{R}_l(t))\, y^l
\end{aligned}$$

where

$$\mathbf{R}_l(t) = \sum_{i=l}^{n-1} \binom{i}{l} a_i \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}.$$

To show that $\omega^{\frac{1}{2}} \mathbf{M_1} \omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}_2(\mathrm{I})$, we need to show that Definition 1.3 is satisfied. Equivalently, we need to present that there exists a $B > 0$ such that

$$\sup_{[x,x+1]\subset \mathrm{I}} \int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})} \right|^2 dt < B$$

where $\mathbf{R}_l(t) \in C^l(\mathrm{I})$ for all $l = 0, 1, \ldots, n-1$ and $0 < \mathbf{S}_\mathrm{l}(\mathrm{t})$ is an auxiliary function in $C^\infty(\mathrm{I})$ satisfying

$$\mathbf{S}_\mathrm{l}(\mathrm{t}) = \mathrm{o}(\mathrm{t}^{\gamma(\mathrm{l}+1)}) \text{ and } \mathbf{S}_\mathrm{l}(\mathrm{t}) = \mathrm{o}(\mathrm{t}^{\gamma(\mathrm{l})}).$$

Note that by AP_1, $a_l(t) \in C^l(\mathrm{I})$ for $l = 0, 1, \ldots, n-1$. Since ω satisfies A_1, we can claim that $\omega \in C^\infty(\mathrm{I})$. This implies that $\omega^{\frac{1}{2}}$ and $\omega^{-\frac{1}{2}}$ are elements of $C^\infty(\mathrm{I})$ by Lemma 2.3. In particular, $\omega \in C^l(\mathrm{I})$ and $\omega^{-\frac{1}{2}} \in C^l(\mathrm{I})$, for $l = 0, 1, \ldots, n-1$. Therefore, $\mathbf{R}_l(t) \in C^l(\mathrm{I})$ for $l = 0, 1, \ldots, n-1$.

Now let $\mathbf{S}_\mathrm{l}(\mathrm{t}) = \omega^{\frac{1}{2}} \mathrm{b}_\mathrm{l} \omega^{-\frac{1}{2}} = \mathrm{b}_\mathrm{l}(\mathrm{t})$. Since $0 < b_l \in C^\infty(\mathrm{I})$, so is $\mathbf{S}_\mathrm{l}$. Clearly, $\mathbf{S}_\mathrm{l}(\mathrm{t}) = \mathrm{o}(\mathrm{t}^{\gamma(\mathrm{l}+1)})$ and $\mathbf{S}_\mathrm{l}(\mathrm{t}) = \mathrm{o}(\mathrm{t}^{\gamma(\mathrm{l})})$ because $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$. Thus, there exists an auxiliary function $\mathbf{S}_\mathrm{l}(\mathrm{t})$.

Next, we evaluate $\left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})} \right|^2$ as follows:

$$\begin{aligned} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})} \right|^2 &= \left| \frac{\sum_{i=l}^{n-1} \binom{i}{l} a_i(t) \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)} \right|^2 \leq J^2 \left| \frac{\sum_{i=l}^{n-1} a_i(t) \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)} \right|^2 \\ &= J^2 \left| \sum_{i=l}^{n-1} \frac{a_i(t)}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right|^2. \end{aligned}$$

where $J = \sup_{l \leq i \leq n-1} \left\{ \binom{i}{l} \right\}$. Using the triangle inequality and (AP_1) we can simplify the above statement as follows, where A is constant.

$$\begin{aligned} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})} \right|^2 &\leq J^2 \sum_{i=l}^{n-1} \left| \frac{a_i(t)}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right|^2 = J^2 \sum_{i=l}^{n-1} \left| \frac{A a_l(t)(t^{i-l})}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right|^2 \\ &= J^2 A^2 \sum_{i=l}^{n-1} \left| \frac{a_l(t)(t^{i-l})}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right|^2 = J^2 A^2 \sum_{i=l}^{n-1} \left| \frac{a_l(t)}{b_l(t)} \frac{t^{i-l} (\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right|^2 \end{aligned}$$

Finally, by using Lemma 2.3 and condition (AP_2) we obtain the following:

$$\begin{aligned}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})}\right|^2 &\leq J^2A^2N^2\sum_{i=l}^{n-1}\left|\frac{a_l(t)}{b_l(t)}\right|^2 = J^2A^2N^2(n-l)^2\left|\frac{a_l(t)}{b_l(t)}\right|^2\\ &\leq J^2A^2N^2(n-l)^2S^2.\end{aligned}$$

Hence, we have shown that

$$\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})}\right|^2 \leq J^2A^2N^2(n-l)^2S^2.$$

This means that there exists a constant $B = J^2A^2N^2(n-l)^2S^2$ such that

$$\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})}\right|^2 \leq B$$

Integrating both sides over the interval $[x, x+1]$, we obtain the following:

$$\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})}\right|^2 dt \leq \int_x^{x+1} B\ dt = B.$$

Consequently, taking the supremum of both sides over the interval $[x, x+1]$ $\subset [1,\infty)$, we have

$$\sup_{[x,x+1]\subset \mathrm{I}}\int_x^{x+1}\left|\frac{\mathbf{R}_l(t)}{\mathbf{S}_\mathrm{l}(\mathrm{t})}\right|^2 dt < B,$$

where $B = J^2A^2N^2(n-l)^2S^2$. Therefore, $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2([1,\infty))$. Furthermore, we have shown that

$$\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}\mathrm{y} = \sum_{\mathrm{l}=0}^{\mathrm{n}-1}\left(\mathbf{R}_\mathrm{l}(\mathrm{t})\right)\mathrm{y}^\mathrm{l}$$

which is obiously an admissible perturbation of the form (10). Surprisingly, $\omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}$ satisfies (11) and 12. We therefore conclude, using Theorem 1.4, that the essential spectrum of $\mathbf{M}_0$ is preserved under this kind of admissible perturbation in $\mathcal{L}^2$-space. In short, we have also proven the following claim:

$$\sigma_e(\mathbf{M}_0 + \omega^{\frac{1}{2}}\mathbf{M_1}\omega^{-\frac{1}{2}}) = \sigma_\mathrm{e}(\mathbf{M}_0).$$

□

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