Symmetric Solutions to a System of Mutually Delay-Coupled Oscillators with Conjugate Coupling

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Abstract. Coupled oscillators are the subject of different studies because they display interesting behavior, such as synchrony. In this paper, we investigate the dynamics of a system of two delay-coupled oscillators and show that they display some kind of synchrony. We use the Stuart-Landau equations to represent this system, which is linked via conjugate coupling. These equations form a system of delay differential equations, which we have found to have symmetry isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We looked into the form of steady-states that inherit part of the symmetry of this system and the birth of periodic solutions with symmetry. In particular, we found bifurcating steady-state and periodic solutions with symmetry isomorphic to $\mathbb{Z}_2$ emanating from the branch of trivial solutions. As an illustration, we performed numerical continuation to generate a bifurcating branch of symmetric solutions from the branch of trivial solutions. These symmetric solutions provide us with the type of synchrony that can be displayed by the oscillators. Finally, we determined the stability of the different branches of symmetric solutions numerically.

1. Introduction

Oscillations are in abundance in nature as well as in manufactured systems and devices. They have been the theme of a good deal of theoretical and experimental studies. Among them are the study of synchrony in coupled oscillators. Research on synchrony traces its origin during the 17^{th} century when a Dutch scientist, Christiaan Huygens, disclosed the synchronization of two pendulum clocks in an experimental set-up [1]. From then on, researches on the emergence of synchrony in coupled systems and the cause of which have been active. Recent studies find the conditions for synchrony in coupled oscillators [2, 3, 4] and discover the appearance of synchrony through different types of coupling [5, 6]. More generally, oscillators that are coupled give venue for observation of collective phenomenon, such as synchronization, in physical and biological systems [7]. Most oscillators can be represented by a universal mathematical model. One of the universal models for oscillators is given by a system of Stuart-Landau (SL) equations. In his paper [8], Verma considered a system of oscillators in a face to face configuration modeled by SL equations with conjugate coupling. In this paper, we modify his model with the addition of a fixed delay on the coupling time. An introduction to delay equations can be found in [4], which is a paper involving a similar face to face configuration of two delay-coupled lasers.

Our main objective in this paper is to classify the symmetric solutions to the system of two oscillators coupled via conjugate coupling. We want to describe the type of synchrony between the oscillators in steady-state and periodic solutions. We show mathematically that

the synchrony of the oscillators is naturally inherited from the symmetry of the equations that describe the two-oscillator system. The mathematical tools used in this paper are group theoretic methods that are found in the books given in [9, 10, 11].

2. Two-Oscillator System with Conjugate Coupling

In this section, we study the symmetric system of two delay-coupled oscillators with conjugate coupling using Stuart-Landau (SL) equations. This system can be represented by the SL equations given below,

$$\begin{array}{rcl} \dot{x_1} & = & (1-x_1^2-y_1^2)x_1 - \omega y_1 + \epsilon(y_2(t-\tau)-x_1) \\ \dot{y_1} & = & (1-x_1^2-y_1^2)y_1 + \omega x_1 \\ \dot{x_2} & = & (1-x_2^2-y_2^2)x_2 - \omega y_2 + \epsilon(y_1(t-\tau)-x_2) \\ \dot{y_2} & = & (1-x_2^2-y_2^2)y_2 + \omega x_2 \end{array}, \tag{1}$$

where τ is the delay in the coupling. The addition of a delay in the coupling is a modification to the system presented by Verma [8]. In his paper, he was able to compute for three forms of steady-state solutions: the trivial solution and two nontrivial solutions having the form $(x, y, -x, -y)$ and (x, y, x, y). In the following, we will also compute for steady-state solutions of (1) but through a group theoretic approach.

2.1. Symmetry Group and Isotropy Subgroups

We claim that system (1) has symmetry isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. To show this, we note that that a homomorphism $\rho : \mathbb{Z}_2 = \langle\gamma\rangle \longrightarrow GL(4, \mathbb{R})$ defined by

$$e \to \mathbb{I}, \quad \gamma \to \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

is a representation of $\mathbb{Z}_2$ on $\mathbb{R}^4$. Here, we see that the action of γ on $[x_1, y_1, x_2, y_2]^T \in \mathbb{R}^4$ is given by $\gamma \cdot [x_1, y_1, x_2, y_2]^T = [x_2, y_2, x_1, y_1]^T$. Also, a homomorphism $\rho : \mathbb{Z}_2 = \langle\sigma\rangle \longrightarrow GL(4, \mathbb{R})$ defined by $e \to \mathbb{I}, \quad \sigma \to -\mathbb{I}$, is a representation of $\mathbb{Z}_2$ on $\mathbb{R}^4$. Here, we see that the action of σ on $[x_1, y_1, x_2, y_2]^T \in \mathbb{R}^4$ is given by $\sigma \cdot [x_1, y_1, x_2, y_2]^T = -[x_1, y_1, x_2, y_2]^T$. Now, if we define $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f([x_1, y_1, x_2, y_2]^T) = [\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2]^T$, then we have that $f(\gamma \cdot [x_1, y_1, x_2, y_2]^T) = \gamma \cdot f([x_1, y_1, x_2, y_2])$ and $f(\sigma \cdot [x_1, y_1, x_2, y_2]^T) = \sigma \cdot f([x_1, y_1, x_2, y_2])$. Therefore, system (1) has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

To find the maximal isotropy subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we start by looking at the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and then at their fixed point subspace. The nontrivial subgroups are $\mathbb{Z}_2(\gamma)$, $\mathbb{Z}_2(\sigma)$, and $\mathbb{Z}_2(\gamma\sigma)$. From here, we get that $\mathrm{Fix}(\mathbb{Z}_2(\gamma)) = \{[x, y, x, y]^T : x, y \in \Re\}$, $\mathrm{Fix}(\mathbb{Z}_2(\sigma)) = \{[0, 0, 0, 0]^T\}$, and $\mathrm{Fix}(\mathbb{Z}_2(\gamma\sigma)) = \{[x, y, -x, -y]^T : x, y \in \Re\}$. Since $[0, 0, 0, 0]^T$ is also fixed by the larger group $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have that the maximal isotropy subgroups are only those given in **Figure 1**.

We give in the following theorem the summary of the above results.

Theorem 1. *$\mathbb{Z}_2(\gamma) \times \mathbb{Z}_2(\sigma)$ is a symmetry group of the system of SL equations given in (1) and its maximal nontrivial isotropy subgroups are $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\gamma\sigma)$.*

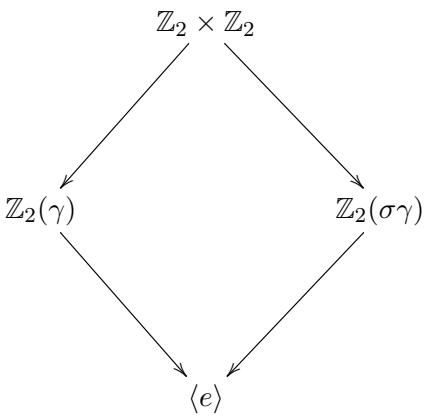

Figure 1. Isotropy subgroup lattice diagram of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.2. Symmetric Steady-States

We now look for symmetric steady-state solutions. These are steady-states that are fixed by a particular isotropy subgroup. The forms of these symmetric steady-state solutions are given in the theorem below.

Theorem 2. *Steady-states with $\mathbb{Z}_2(\gamma)$ symmetry of the system given in (1) can be found by solving for $y = \pm\frac{\sqrt{-\epsilon(a-2\omega-\epsilon)(a+2-\epsilon)}}{\epsilon(a-2\omega-\epsilon)}\omega$ and $x = \frac{1}{2\omega}(-\epsilon + a)y$ or $y = \pm\frac{\sqrt{-\epsilon(a+2\omega+\epsilon)(a-2+\epsilon)}}{\epsilon(a+2\omega+\epsilon)}\omega$ and $x = \frac{1}{2\omega}(-\epsilon - a)y$, where $a = \sqrt{\epsilon^2 - 4\omega^2 + 4\omega\epsilon}$. Also, steady-states with $\mathbb{Z}_2(\gamma\sigma)$ symmetry can be found by solving for $y = \pm\frac{\sqrt{-\epsilon(b+2\omega-\epsilon)(b+2-\epsilon)}}{\epsilon(b+2\omega-\epsilon)}\omega$ and $x = \frac{1}{2\omega}(-\epsilon + b)y$ or $y = \pm\frac{\sqrt{-\epsilon(b-2\omega+\epsilon)(b-2+\epsilon)}}{\epsilon(b-2\omega+\epsilon)}\omega$ and $x = \frac{1}{2\omega}(-\epsilon - b)y$, where $b = \sqrt{\epsilon^2 - 4\omega^2 - 4\omega\epsilon}$. Moreover, the birth of steady-states with $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$ symmetry along the branch of trivial solutions for different values of ϵ is at $\epsilon = (1+\omega^2)/(1+\omega)$ and $\epsilon = (1+\omega^2)/(1-\omega)$, respectively.*

proof :

For steady-states with $\mathbb{Z}_2(\gamma)$ symmetry (steady-states of the form $[x, y, x, y]^T$), we solve the system $\begin{cases} 0 &= (1-x^2-y^2)x - \omega y + \epsilon(y-x) \\ 0 &= (1-x^2-y^2)y + \omega x \end{cases}$. This is equivalent to solving $0 = (-\omega)x^2 - (\epsilon y)x + (\epsilon - \omega)y^2$, $y \neq 0$. Solving for x using the quadratic formula, we get that $x = \frac{1}{2\omega}(-\epsilon \pm a)y$, where $a = \sqrt{\epsilon^2 - 4\omega^2 + 4\omega\epsilon}$. For $x = \frac{1}{2\omega}(-\epsilon + a)y$, we get $y = \pm\frac{\sqrt{-\epsilon(a-2\omega-\epsilon)(a+2-\epsilon)}}{\epsilon(a-2\omega-\epsilon)}\omega$ and for $x = \frac{1}{2\omega}(-\epsilon - a)y$, we get $y = \pm\frac{\sqrt{-\epsilon(a+2\omega+\epsilon)(a-2+\epsilon)}}{\epsilon(a+2\omega+\epsilon)}\omega$. Now, to determine the birth of steady-states with $\mathbb{Z}_2(\gamma)$ symmetry along the branch of trivial solutions for different values of ϵ, we solve for ϵ such that $x = 0$ and $y = 0$ in the formulas in the previous line. Here, we get that $\epsilon = (1+\omega^2)/(1+\omega)$.

Similarly, for steady-states with $\mathbb{Z}_2(\sigma\gamma)$ symmetry (steady-states of the form $[x, y, -x, -y]^T$), we solve the system $\begin{cases} 0 &= (1-x^2-y^2)x - \omega y + \epsilon(-y-x) \\ 0 &= (1-x^2-y^2)y + \omega x \end{cases}$, which is equivalent to solving $0 = (-\omega)x^2 + (-\epsilon y)x - (\epsilon + \omega)y^2$, $y \neq 0$. Solving for x, we get $x = \frac{1}{2\omega}(-\epsilon \pm b)y$, where $b = \sqrt{\epsilon^2 - 4\omega^2 - 4\omega\epsilon}$. For $x = \frac{1}{2\omega}(-\epsilon + b)y$, we get $y = \pm\frac{\sqrt{-\epsilon(b+2\omega-\epsilon)(b+2-\epsilon)}}{\epsilon(b+2\omega-\epsilon)}\omega$ and for $x = \frac{1}{2\omega}(-\epsilon - b)y$, we get $y = \pm\frac{\sqrt{-\epsilon(b-2\omega+\epsilon)(b-2+\epsilon)}}{\epsilon(b-2\omega+\epsilon)}\omega$. Also, we have that the birth of steady-states with $\mathbb{Z}_2(\sigma\gamma)$ symmetry along the branch of trivial solutions is at $\epsilon = (1+\omega^2)/(1-\omega)$, which does not exist for $\omega \geq 1$. ■

2.3. Linearization and Block Diagonalization

At this point, we classify the symmetry of bifurcation points according to which block the critical eigenvalue comes from. The classification is given by the succeeding theorem.

Theorem 3. *If* $V_1 = span\{[1,0,1,0]^T\}$, $V_2 = span\{[0,1,0,1]^T\}$, $V_3 = span\{[1,0,-1,0]^T\}$, $V_4 = span\{[0,1,0,-1]^T\}$, *then* $\mathbb{Z}_2(\gamma)$ *and* $\mathbb{Z}_2(\sigma\gamma)$ *lie in the isotropy subgroup of solutions (of (1) near* $[0,0,0,0]^T$*) coming from* $V_1 \oplus V_2$ *and* $V_3 \oplus V_4$, *respectively.*

proof :

We first linearize system (1) around $[0,0,0,0]^T$, which is the steady-state solution with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. To do this, we note that system (1) can be seen as a function of the variables with delay and those without delay, that is, we can express system (1) into the form $\dot{Z}(t) = F(Z(t), Z(t-\tau))$. To get the linear variational equation around the trivial steady-state, we compute $d_{Z(t)}F$ and $d_{Z(t-\tau)}F$ then evaluate them at $O := [0,0,0,0]^T$. We get

$$M_1 := d_{Z(t)}F(O) = \begin{bmatrix} 1-\epsilon & -\omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & 1-\epsilon & -\omega \\ 0 & 0 & \omega & 1 \end{bmatrix}, \; M_2 := d_{Z(t-\tau)}F(O) = \begin{bmatrix} 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the linear variational equation around $[0,0,0,0]^T$ is $\dot{Z}(t) = M_1 Z(t) + M_2 Z(t-\tau)$. The characteristic equation of this linear variational equation is given by $\det \Delta(\lambda) = 0$. Here,

$$L := \Delta(\lambda) = \lambda I - M_1 - e^{-\lambda\tau} M_2 = \begin{bmatrix} A & B & 0 & D \\ -B & C & 0 & 0 \\ 0 & D & A & B \\ 0 & 0 & -B & C \end{bmatrix},$$

where $A = \lambda - 1 + \epsilon$, $B = \omega$, $C = \lambda - 1$, and $D = -\epsilon\, e^{-\lambda\tau}$. Now, our objective is to reduce L into block diagonal form. In block diagonal form, we will be able to solve for eigenvalues by dealing with smaller matrices and we can characterize the bifurcations depending on which block the eigenvalue comes from. We begin by noting that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts irreducibly on the subspaces V_1, V_2, V_3, and V_4 with basis elements $v_1 = v \cdot [1,0,1,0]^T$, $v_2 = v \cdot [0,1,0,1]^T$, $v_3 = v \cdot [1,0,-1,0]^T$, and $v_4 = v \cdot [0,1,0,-1]^T$. respectively. Observe that $L(v_1) = Av_1 + (-B)v_2$ and $L(v_2) = (B+D)v_1 + Cv_2$. Therefore, $L(v_1 + v_2) = E_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $E_1 := \begin{bmatrix} A & -B \\ B+D & C \end{bmatrix}$. Also, we have that $L(v_3 + v_4) = E_2 \begin{bmatrix} v_3 \\ v_4 \end{bmatrix}$ where $E_2 := \begin{bmatrix} A & -B \\ B-D & C \end{bmatrix}$. Hence, in block diagonal form, L is given by $L = \Delta(\lambda) = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$ and the eigenvalues of $L|V_1 \oplus V_2$ and $L|V_3 \oplus V_4$ are those of E_1 and E_2, respectively. This block diagonalization technique is patterned from methods in equivariant theory given in [12].

Note that the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ breaks up $\mathbb{R}^4$ into $\mathbb{R}^2 \oplus \mathbb{R}^2$, where $\mathbb{R}^2 = V_1 \oplus V_2$ and $\mathbb{R}^2 = V_3 \oplus V_4$, respectively. As observed in the block diagonalization process, the block E_1 corresponds to the action of L on $\mathbb{R}^2 = V_1 \oplus V_2$ and the block E_2 corresponds to the action of L on $\mathbb{R}^2 = V_3 \oplus V_4$. If the critical eigenvalue λ comes from the block E_1, we get bifurcations with symmetry $\mathbb{Z}_2(\gamma)$ because $\mathbb{Z}_2(\gamma)$ acts trivially on $V_1 \oplus V_2$. To say it in another way, $\mathbb{Z}_2(\gamma)$ lies in the isotropy subgroup of solutions coming from $V_1 \oplus V_2$. If the critical eigenvalue λ comes from the block E_2, we get bifurcations with symmetry $\mathbb{Z}_2(\sigma\gamma)$ because $\mathbb{Z}_2(\sigma\gamma)$ acts trivially on $V_3 \oplus V_4$. Similarly, $\mathbb{Z}_2(\sigma\gamma)$ lies in the isotropy subgroup of solutions coming from $V_3 \oplus V_4$. ■

2.4. Symmetry-Breaking Bifurcation Points

Here, we find symmetry-breaking steady-state and Hopf bifurcation points along the branch of trivial solutions, which is obtained by varying ϵ. That is, ϵ is our bifurcation parameter. Since the $\Delta(\lambda)$ is a block diagonal matrix, the eigenvalues can be found by solving for λ in equations $\det(E_1) = 0$ and $\det(E_2) = 0$. We have that, $|E_1| = (\lambda - 1 + \epsilon)(\lambda - 1) - (\omega - \epsilon\ e^{-\lambda\tau})(-\omega) = 0$ and $|E_2| = (\lambda - 1 + \epsilon)(\lambda - 1) - (\omega + \epsilon\ e^{-\lambda\tau})(-\omega) = 0$.

For steady-state bifurcation points, the critical eigenvalue λ is zero. Therefore, we find steady state bifurcation points by evaluating the determinant at the eigenvalue $\lambda = 0$ and then solving for ϵ. We do that for E_1 and we get $\epsilon = \frac{1+\omega^2}{1+\omega}$. For E_2, we get $\epsilon = \frac{1+\omega^2}{1-\omega}$. Observe that the steady-state bifurcation points from E_1 and E_2 are also the point of birth of steady-state solutions with $\mathbb{Z}_2(\gamma)$ symmetry and $\mathbb{Z}_2(\sigma\gamma)$ symmetry along the branch of trivial solutions, respectively.

For Hopf bifurcation points, the critical eigenvalue λ is a pair of imaginary numbers. Therefore, we find Hopf bifurcation points by evaluating the determinant at the eigenvalue $\lambda = i\beta$ and then solving for ϵ and β. For $|E_1|_{\lambda=i\beta} = 0$, we get a complex equation that we separate into real and imaginary parts. We get, $-\beta^2 + 1 - \epsilon + \omega^2 - \omega\epsilon\cos(\beta\tau) = 0$ and $\beta(2 - \epsilon) - \omega\epsilon\sin(\beta\tau) = 0$. These are two equations in the variables ϵ and β. We solve for the ϵ and β values that satisfy these two equations by looking for their intersections using a computer algebra package. Similarly, for $|E_2|_{\lambda=i\beta} = 0$, we get, $-\beta^2 + 1 - \epsilon + \omega^2 + \omega\epsilon\cos(\beta\tau) = 0$ and $-\beta(2 - \epsilon) - \omega\epsilon\sin(\beta\tau) = 0$.

As an example, we solve for values of ϵ where we get bifurcation points for given values of ω and τ.

Steady-State Bifurcation Points		Hopf Bifurcation Points	
$\mathbb{Z}_2(\gamma)$ symmetry	$\mathbb{Z}_2(\sigma\gamma)$ symmetry	$\mathbb{Z}_2(\gamma)$ symmetry	$\mathbb{Z}_2(\sigma\gamma)$ symmetry
$\epsilon = 0.833333$	$\epsilon = 2.5$	None	$\epsilon = 2.105227$ $\beta = 0.443118$

Table 1. Bifurcation points when $\omega = 0.5$ and $\tau = 0.1$

2.5. Branches of Symmetric Steady-States and Periodic Solutions

Finally, we use DDE-Biftool [13] to get branches of symmetric steady-state and periodic solutions. As an example, we plot the symmetric steady states, computed using Theorem 2, along the generated branches. We also plot the symmetry breaking bifurcation points and illustrate the classification of symmetric solutions based on which block the critical eigenvalue comes from.

We note that the trivial solution is the only steady-state solution with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. We use DDE-Biftool to follow the trivial solution into a branch of trivial solutions by varying ϵ. We observe from Figure 1 that from a branch of solutions with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, we can expect a bifurcating branch of symmetric solutions with symmetry $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$. Also, we have shown in subsection 2.3 and 2.4 that we can solve for symmetry-breaking bifurcation points along the branch of trivial solutions.

We now generate a branch of trivial steady-state solutions using the following parameter values: $\omega = 0.5$, $\tau = 0.1$, and ϵ is varied from 0 to 6. Furthermore, we plot along this branch the steady-state and Hopf bifurcation points that were solved in Table 1. In DDE-BIftool, we follow the steady state bifurcation points to get a bifurcating branch of steady-state solutions with symmetry $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$. We also follow the Hopf bifurcation point to get a branch of periodic solutions with symmetry $\mathbb{Z}_2(\sigma\gamma)$. Note that for these parameter values, there is no Hopf bifurcation point with $\mathbb{Z}_2(\gamma)$ symmetry. We give the plot in Figure 2 (Left). The profile of the periodic solutions is given in Figure 2 (Right).

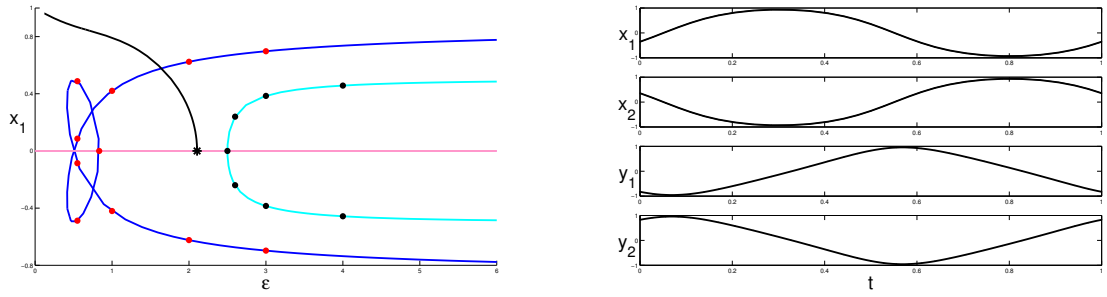

Figure 2. (Left) Branch of trivial steady-states in the $(\epsilon, x_1)-$plane (pink). The red dot and black dot along the branch of trivial steady-states correspond to the steady-state bifurcation point with symmetry $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$, respectively. The black asterisk corresponds to Hopf bifurcation point with $\mathbb{Z}_2(\sigma\gamma)$ symmetry. The branch of steady-states with $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$ symmetry are colored blue and cyan, respectively. The red and black dots are steady-state solutions that are computed using Theorem 2. The branch of periodic solutions with $\mathbb{Z}_2(\sigma\gamma)$ symmetry is colored black. (Right) The profile of the periodic solutions in Figure 2 (Left) shows that $x_1 = -x_2$ and $y_1 = -y_2$. This illustrates that the periodic solutions emanating from Hopf bifurcations with $\lambda = i\beta$ coming from E_2 have symmetry $\mathbb{Z}_2(\sigma\gamma)$.

We also perform the above process for the parameters: $\omega = 3$, $\tau = 0.1$, and ϵ is varied from 0 to 6 (see Figures 3 and 4). Here, we are not able to get a branch of steady-state solutions with $\mathbb{Z}_2(\sigma\gamma)$ symmetry.

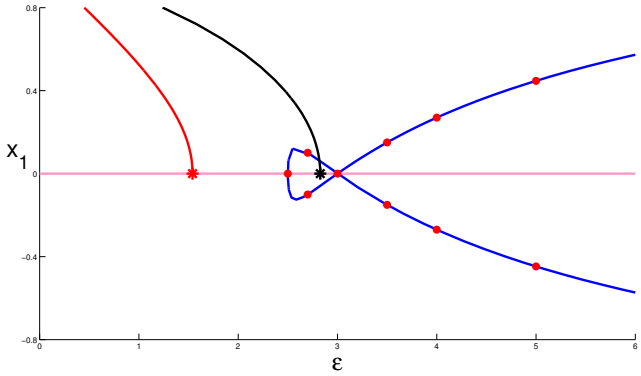

Figure 3. Branch of Trivial Steady States in the $(\epsilon, x_1)-$plane (pink). The branch in blue corresponds to a branch of steady-state solutions with $\mathbb{Z}_2(\gamma)$ symmetry and the red dots are steady-states that were computed using Theorem 2. The branch in red is the branch of periodic solutions with $\mathbb{Z}_2(\gamma)$ symmetry emanating from the Hopf bifurcation point, which is computed with $\lambda = i\beta$ coming from E_1. The branch in black is the branch of periodic solutions with $\mathbb{Z}_2(\sigma\gamma)$ symmetry emanating from the Hopf bifurcation point, which is computed with $\lambda = i\beta$ coming from E_2.

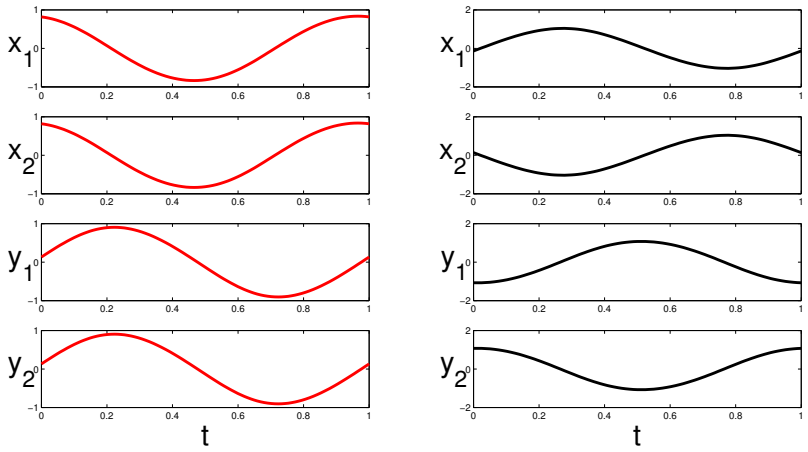

Figure 4. The profile of the periodic solutions in Figure 3 corresponding to the branch in red and black, respectively. These profiles illustrate the $\mathbb{Z}_2(\gamma)$ and $\mathbb{Z}_2(\sigma\gamma)$ symmetry of periodic solutions emanating from Hopf bifurcations with $\lambda = i\beta$ coming from E_1 and E_2, respectively.

Now, DDE-Biftool can also determine the branch stability of different solutions. We utilize this capability of DDE-Biftool to determine the stability of the branches of symmetric solutions that are given in Figures 2 and 3. The resulting stability of the different branches are given in Figure 5.

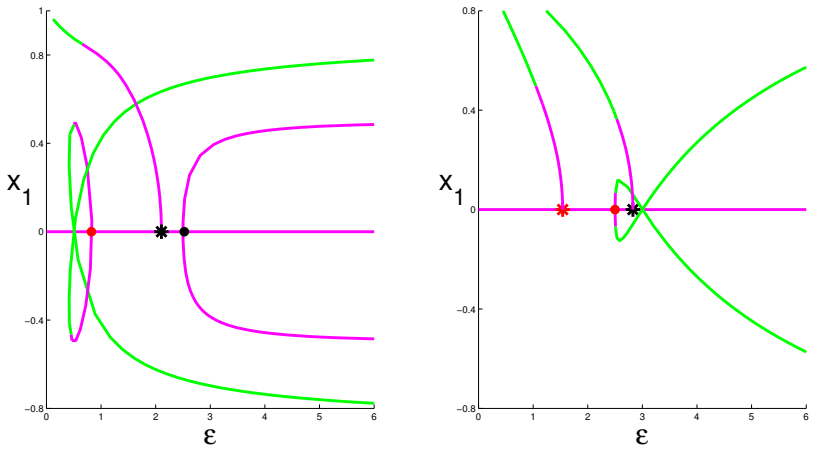

Figure 5. The stability of the branches of symmetric solutions given in Figures 2 and 3. The stable part of the branch is in green while the unstable part is in magenta.

3. Conclusion

The main result of this paper is the classification of the symmetry of solutions to the system of mutually delay-coupled oscillators with conjugate coupling modeled via Stuart-Landau equations, that is, the types of synchrony that can be attained by the oscillators. We found the types of synchrony of steady-state solutions by identifying the nontrivial isotropy subgroups of the symmetry group of the two-oscillator system. Specifically, we found that the steady-states

can have symmetry that are isomorphic to $\mathbb{Z}_2$. Also, we used the isotypic decomposition of the physical space to block diagonalize the characteristic matrix of the linear variational equation around the trivial steady-state solution. The isotypic decomposition is then utilized to classify the type of synchrony of steady-state and Hopf bifurcations based on which diagonal block the critical eigenvalue came from. Here, the symmetry of steady-state and Hopf bifurcations were found to be isomorphic to $\mathbb{Z}_2$. The block diagonalization also simplifies the computation of steady-state and Hopf bifurcation points. As an illustration, we did numerical continuation and analysis using DDE-Biftool to generate branches of symmetric steady-state and periodic solutions along the branch of trivial steady-state solutions and to determine the stability of these branches.

References

[1] Pikovsky A and Rosenblum M 2007 *Scholarpedia* **2** 1459
[2] Goel P and Ermentrout B 2002 *Physica D* **163** 191–216
[3] D'Huys O and etal 2010 *Proc. of SPIE* **7720** 772023
[4] Collera J 2015 *International Journal of Philippine Science and Technology* **08** 17–21
[5] Tajima S and et al 2016 *Journal of Electroanalytical Chemistry* **769** 16–20
[6] Poel W, Zakharova A and Scholl E 2015 *Physical Review E* **91** 022915
[7] Zoo W and et al 2013 *Physical Review Letters* **111** 014101
[8] Verma U 2011 Emergent states of limit cycle oscillators coupled via conjugate variable, CIMPA Poster
[9] Chossat P and Lauterbach R 2000 *Methods in Equivariant Bifurcations and Dynamical Systems* (World Scientific Publishing Co. Pte. Ltd.)
[10] Golubitsky M and Stewart I 2002 *The Symmetry Perspective* (Birkhauser Verlag)
[11] Field M 2007 *Dynamics and Symmetry* (Imperial College Press)
[12] Golubitsky M, Stewart I and Schaeffer D 1988 *Singularities and Groups in Bifurcation Theory* vol II (Springer-Verlag)
[13] Engelborghs K, Luzyanina T and Samaey G 2001 DDE-BIFTOOL v. 2.00 User manual: A matlab package for bifurcation analysis of delay differential equations Tech. rep. Department of Computer Science, K. U. Leuven, Leuven