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Factorizations of Lorentz Matrices

Edna N. Gueco
Department of Mathematics and Computer Science
University of the Philippines Baguio
Baguio City, Philippines
edna.gueco@upb.edu.ph

Dennis I. Merino
Department of Mathematics
Southeastern Louisiana University
Hammond, LA, USA
dmerino@selu.edu

Agnes T. Paras
Institute of Mathematics
University of the Philippines Diliman
Quezon City, Philippines
agnes@math.upd.edu.ph

Abstract

For a nonnegative integer $k \leq n$, let $L_k \equiv I_k \oplus -I_{n-k}$. An $A \in M_n(\mathbb{C})$ is called *Lorentz* if $L_k A^* L_k = A^{-1}$. Let $\mathcal{O}_{L_k}$ be the set of all Lorentz matrices. We show that every $A \in \mathcal{O}_{L_k}$ has a polar decomposition $A = PU$, where P is positive definite, U is unitary, and both P and U are in $\mathcal{O}_{L_k}$. We also show that for every $A \in \mathcal{O}_{L_k}$, there exist nonnegative integers r and s, unitary $P, Q \in \mathcal{O}_{L_k}$, and positive definite diagonal $D, E \in M_{k-r}(\mathbb{C})$ such that $k - r = n - k - s$, $D^2 - E^2 = I_{k-r}$, and $A = P\left(I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s\right)Q^*$.

1 Introduction

Our notation is standard as in [3, 4]. We let $M_n(\mathbb{C})$ be the set of all n-by-n complex matrices, and we let $\mathcal{H}_n$ be the set of all nonsingular Hermitian $H \in M_n(\mathbb{C})$. Let $S \in M_n(\mathbb{C})$ be nonsingular. Define $\Lambda_S : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\Lambda_S(A) = S^{-1}A^*S$. One checks that for every $A, B \in M_n(\mathbb{C})$, we have $\Lambda_S(AB) = \Lambda_S(B)\Lambda_S(A)$. In particular, for every positive integer k, we have $\Lambda_S\left(A^k\right) = (\Lambda_S(A))^k$. Now, for every $\alpha, \beta \in \mathbb{C}$, we have $\Lambda_S(\alpha A + \beta B) = \overline{\alpha}\Lambda_S(A) + \overline{\beta}\Lambda_S(B)$. Hence, if $p(x)$ is a polynomial with real coefficients, then $\Lambda_S(p(A)) = p(\Lambda_S(A))$.

One checks that $\Lambda_S(A) = I$ if and only if $A = I$. It follows that if A is nonsingular, then $(\Lambda_S(A))^{-1} = \Lambda_S\left(A^{-1}\right)$. Notice that $\Lambda_S(\Lambda_S(A)) = S^{-1}S^*AS^{-*}S$. Hence, if $S \in \mathcal{H}_n$, then for every $A \in M_n(\mathbb{C})$, we have $\Lambda_S(\Lambda_S(A)) = A$.

Lemma 1. *Let $S \in M_n(\mathbb{C})$ be nonsingular. Then $\Lambda_S(\Lambda_S(A)) = A$ for every $A \in M_n(\mathbb{C})$ if and only if there exists $H \in \mathcal{H}_n$ such that $\Lambda_S = \Lambda_H$.*

Proof. The converse has just been shown. For the forward implication, suppose that $\Lambda_S(\Lambda_S(A)) = A$ for every $A \in M_n(\mathbb{C})$. Then, we have $S^{-1}S^*A = AS^{-1}S^*$, so that there exists a nonzero $\alpha \in \mathbb{C}$ such that $S^{-1}S^* = \alpha I$. Hence, $S^* = \alpha S$ or $S = \frac{1}{\alpha}S^*$. Observe now that $\alpha S = S^* = \left(\frac{1}{\alpha}S^*\right)^* = \frac{1}{\overline{\alpha}}S$, so that $\alpha\overline{\alpha}S = S$. Thus, $|\alpha| = 1$ and there exists $\theta \in \mathbb{R}$ such that $\alpha = e^{i\theta}$. Set $H \equiv e^{\frac{i\theta}{2}}S$ and notice that $H^* = e^{-\frac{i\theta}{2}}S^* = e^{\frac{i\theta}{2}}S = H$ so that $H \in \mathcal{H}_n$. Moreover, $\Lambda_H(A) = H^{-1}A^*H = e^{-\frac{i\theta}{2}}S^{-1}A^*e^{\frac{i\theta}{2}}S = S^{-1}A^*S = \Lambda_S(A)$. □

Let $U \in M_n(\mathbb{C})$ be unitary. Then $U^* = U^{-1}$. Notice that $U^* = \Lambda_H(U)$ when $H = I$.

Definition 2. *Let $H \in \mathcal{H}_n$ be given. A given $B \in M_n(\mathbb{C})$ is called* Λ_H orthogonal *if B is nonsingular and $\Lambda_H(B) = B^{-1}$.*

Let $H \in \mathcal{H}_n$ be given. A Λ_H orthogonal matrix is also called $H-$*unitary* [6]. We denote by $\mathcal{O}_H$ the set of all Λ_H orthogonal matrices. Let $A, B \in \mathcal{O}_H$ be given. Then $\Lambda_H(AB) = \Lambda_H(B)\Lambda_H(A) = B^{-1}A^{-1} = (AB)^{-1}$, so that $\mathcal{O}_H$ is a group under multiplication. We study $\mathcal{O}_H$. Now, for given $H, G \in \mathcal{H}_n$, there exists a nonsingular $X \in M_n(\mathbb{C})$ such that $H = X^*GX$ if and only if the inertia of H (number of positive and negative eigenvalues) is the same as the inertia of G. For such H and G, we have $A^{-1} = H^{-1}A^*H$ if and only if $\left(XAX^{-1}\right)^{-1} = \left(XH^{-1}X^*\right)\left(X^{-*}A^*X^*\right)\left(X^{-*}HX^{-1}\right) = G^{-1}\left(XAX^{-1}\right)^*G$. Thus, $A \in \mathcal{O}_H$ if and only if $XAX^{-1} \in \mathcal{O}_G$,

Lemma 3. *Let $H, G \in \mathcal{H}_n$ be given. Suppose that H and G have the same inertia, so that there exists a nonsingular $X \in M_n(\mathbb{C})$ such that $H = X^*GX$. Then $A \in \mathcal{O}_H$ if and only if $XAX^{-1} \in \mathcal{O}_G$.*

Let $H \in \mathcal{H}_n$ be given. There exist a nonnegative integer k and a nonsingular $X \in M_n(\mathbb{C})$ such that $L_k \equiv I_k \oplus -I_{n-k}$ and $H = X^*L_kX$. Notice that $L_k \in \mathcal{H}_n$. An $A \in \mathcal{O}_{L_k}$ is also called a *Lorentz* matrix [1, 2, 6]. It is a generalization of unitary matrices to the indefinite product induced by L_k. We note that there are two such indefinite products. Let $x = [x_i] \in \mathbb{C}^n$ be given. One such indefinite product is given by $\langle x, x\rangle_{1,L_k} = x^TL_kx = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$. The other indefinite product is given by $\langle x, x\rangle_{2,L_k} = x^*L_kx = |x_1|^2 + \cdots + |x_k|^2 - |x_{k+1}|^2 - \cdots - |x_n|^2$. We are interested in the second indefinite product. We show that every $A \in \mathcal{O}_{L_k}$ has a polar decomposition $A = PU$, where P is positive definite, U is unitary, and both P and U are in $\mathcal{O}_{L_k}$. We also show that for every $A \in \mathcal{O}_{L_k}$, there exist nonnegative integers r and s, unitary $P, Q \in \mathcal{O}_{L_k}$, and positive definite diagonal $D, E \in M_{k-r}(\mathbb{C})$ such that $k - r = n - k - s$, $D^2 - E^2 = I_{k-r}$, and $A = P\left(I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s\right)Q^*$.

2 Lorentz Matrices

Let an integer $0 \le k \le n$ be given. Let $U_1 \in M_k(\mathbb{C})$ and let $U_2 \in M_{n-k}(\mathbb{C})$ be both unitary. Then $B \equiv U_1 \oplus U_2 \in \mathcal{O}_{L_k}$. Conversely, suppose that $C \in \mathcal{O}_{L_k}$ is unitary. Then $L_kC^{-1}L_k = L_kC^*L_k = \Lambda_{L_k}(C) = C^{-1}$. Consequently, $L_kC = CL_k$. Writing

$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, conformal to L_k, one checks that $C_{12} = 0$ and that $C_{21} = 0$. Since C is unitary, both C_{11} and C_{22} are unitary.

Proposition 4. *Let $C \in \mathcal{O}_{L_k}$ be given. Then C is unitary if and only if there exist unitary $C_1 \in M_k(\mathbb{C})$ and $C_2 \in M_{n-k}(\mathbb{C})$ such that $C = C_1 \oplus C_2$.*

The set of unitary matrices in $\mathcal{O}_{L_k}$ is a subgroup of $\mathcal{O}_{L_k}$ under multiplication. Our approach is to study normal ($AA^* = A^*A$) matrices in $\mathcal{O}_{L_k}$. Normal matrices are unitarily diagonalizable. Moreover, if X and $Y \in M_n(\mathbb{C})$ are normal, then X is similar to Y if and only if X is similar to Y with a unitary matrix as a matrix of similarity. Suppose that B and C are in $\mathcal{O}_{L_k}$ and are both normal. If B is similar to C, can the matrix of similarity be taken to be a unitary in $\mathcal{O}_{L_k}$?

Consider $L_1 = \operatorname{diag}(1, -1)$. Let $A = L_1$ and let $B = -L_1$. Then A and B are both Λ_{L_1} orthogonal. Moreover, A is similar to B. The set of unitary matrices in $\mathcal{O}_{L_1}$ is $\{\operatorname{diag}(e^{i\theta}, e^{i\beta}) : \theta, \beta \in \mathbb{R}\}$. Hence, A and B cannot be similar using a unitary matrix in $\mathcal{O}_{L_1}$.

The form of a unitary matrix in $\mathcal{O}_{L_k}$ gives rise to the following definitions.

Let $x = [x_i] \in \mathbb{C}^n$ be given. Suppose that $0 < k < n$. If $x_1 = \cdots = x_k = 0$, then we call x a $(0, n-k)$ *vector.* If $x_{k+1} = \cdots = x_n = 0$, then we call x a $(k, 0)$ *vector.* We let $\mathbb{A}^k$ be the set of all $(k, 0)$ vectors, and we let $\mathbb{B}^{n-k}$ be the set of all $(0, n-k)$ vectors. Notice that for every $y \in \mathbb{C}^n$, we have $y + L_k y \in \mathbb{A}^k$, while $y - L_k y \in \mathbb{B}^{n-k}$. Each of $\mathbb{A}^k$ and $\mathbb{B}^{n-k}$ is a subspace of $\mathbb{C}^n$.

Let $A \in M_n(\mathbb{C})$ be normal. Then $x \in \mathbb{C}^n$ is an eigenvector of A corresponding to λ if and only if x is an eigenvector of A^* corresponding to $\overline{\lambda}$. This follows from the fact that A is unitarily diagonalizable. If $x \in \mathbb{C}^n$ is an eigenvector of A corresponding to λ, if $y \in \mathbb{C}^n$ is an eigenvector of A corresponding to β, and if $\lambda \neq \beta$, then $x^* y = 0$, that is, x and y are orthogonal. For λ an eigenvalue of A, set $V_\lambda \equiv \{x \in \mathbb{C}^n : Ax = \lambda x\}$. Notice that V_λ is a vector space. Let $\dim(V_\lambda)$ denote the dimension of V_λ. If λ and β are distinct eigenvalues of A, then V_λ and V_β are orthogonal. If $x \in V_\lambda$, then $A(Ax) = A(\lambda x) = \lambda(Ax)$ so that $Ax \in V_\lambda$. Suppose that $Ax \in V_\lambda$. If A is singular, then x need not be in V_λ – take $z \in V_\lambda$, let $0 \neq y$ be such that $Ay = 0$, and let $x = z + y$ Then $Ax = Az \in V_\lambda$, however, there is no guarantee that $x \in V_\lambda$. If A is nonsingular, however, then $A(Ax) = \lambda(Ax)$ implies that $Ax = \lambda x$ and $x \in V_\lambda$. Let $V_\lambda^\perp \equiv \{y \in \mathbb{C}^n : y^* x = 0 \text{ for every } x \in V_\lambda\}$ be the *orthogonal complement* of V_λ in $\mathbb{C}^n$. Then, $\operatorname{span}\{V_\lambda, V_\lambda^\perp\} = \mathbb{C}^n$, that is, for every $z \in \mathbb{C}^n$, there exist $x \in V_\lambda$ and $y \in V_\lambda^\perp$ such that $z = x + y$. Let $y \in V_\lambda^\perp$ be given. Write $Ay = u + v$, with $u \in V_\lambda$ and $v \in V_\lambda^\perp$. Then $0 = \left(\overline{\lambda} u\right)^* y = (A^* u)^* y = u^* A y = u^*(u + v) = u^* u$ so that $u = 0$. Hence, if $y \in V_\lambda^\perp$, then $Ay \in V_\lambda^\perp$. Let λ and β be distinct eigenvalues of A. Set $V = \operatorname{span}\{V_\lambda, V_\beta\}$, and let $V^\perp$ be the orthogonal complement of V in $\mathbb{C}^n$. Let $x \in V$ be given. There exist $y \in V_\lambda$ and $z \in V_\beta$ such that $x = y + z$. Notice that $Ay \in V_\lambda$ and that $Az \in V_\beta$, hence, $Ax \in V$. Consequently, if $x \in V^\perp$, then $Ax \in V^\perp$.

Lemma 5. *Let $A \in \mathcal{O}_{L_k}$ be normal. Then $x \in \mathbb{C}^n$ is an eigenvector of A corresponding to the eigenvalue λ if and only if $L_k x$ is an eigenvector of A corresponding to the eigenvalue $\frac{1}{\overline{\lambda}}$.*

Proof. Let $x \in \mathbb{C}^n$ be an eigenvector of A corresponding to λ. Then $Ax = \lambda x$ and we also have $A^* x = \overline{\lambda} x$. Hence, $\overline{\lambda} L_k x = L_k A^* x = L_k A^* L_k L_k x = A^{-1} L_k x$, so that $A L_k x = \frac{1}{\overline{\lambda}} L_k x$. The other direction can be shown by setting $y = L_k x$ and $\beta = \frac{1}{\overline{\lambda}}$, and noting that $L_k y = x$ and $\frac{1}{\overline{\beta}} = \lambda$. $\square$

Let $\mathcal{A} = \{x_1, ..., x_t\}$ be an orthonormal set. Then $(L_k x_p)^* (L_k x_q) = x_p^* x_q$ for every $p, q = 1, ..., t$. Hence, $\mathcal{B} = \{L_k x_1, ..., L_k x_t\}$ is also an orthonormal set. For $p = 1, ..., t$, set $y_p = x_p + L_k x_p$, set $z_p = x_p - L_k x_p$, and set

$$\mathcal{C} = \{y_1, ..., y_t, z_1, ..., z_t\}. \tag{1}$$

Then $x_p = \frac{1}{2}(y_p + z_p)$ and $L_k x_p = \frac{1}{2}(y_p - z_p)$ so that span$(\mathcal{C})$ = span$(\mathcal{A} \cup \mathcal{B})$. Notice that each $y_p \in \mathbb{A}^k$, while each $z_j \in \mathbb{B}^{n-k}$.

Suppose that $|\lambda| = 1$. Then Lemma 5 guarantees that $x \in V_\lambda$ if and only if $L_k x \in V_\lambda$. Moreover, because V_λ is a vector space, $x + L_k x$ and $x - L_k x$ are also in V_λ. Let $\mathcal{A}$ be an orthonormal basis of V_λ and let $\mathcal{C}$ be as in Eq. (1). Then, $V_\lambda = $ span$(\mathcal{C})$. Let $\mathcal{E}$ be a maximal linearly independent subset of $\mathcal{C}$. Then $\mathcal{E}$ is a basis of V_λ. Write $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ contains only $(k, 0)$ vectors and $\mathcal{E}_2$ contains only $(0, n-k)$ vectors. Let $\mathcal{F}_1 = \{f_{11}, ..., f_{1p}\}$ be an orthonormal basis of span$(\mathcal{E}_1)$ and let $\mathcal{F}_2 = \{f_{21}, ..., f_{2q}\}$ be an orthonormal basis of span$(\mathcal{E}_2)$. Then $\mathcal{F}_1 \cup \mathcal{F}_2$ is an orthonormal basis for V_λ consisting only of vectors from $\mathbb{A}^k \cup \mathbb{B}^{n-k}$.

Lemma 6. *Let $A \in \mathcal{O}_{L_k}$ be normal. Let $\lambda_1, ..., \lambda_t$ be the distinct eigenvalues of A with $|\lambda_1| = \cdots = |\lambda_t| = 1$. Then, for each $p, q = 1, ...t$ if $p \neq q$, then V_{λ_p} and V_{λ_q} are orthogonal. For each $p = 1, ..., t$, V_{λ_p} has an orthonormal basis consisting only of vectors from $\mathbb{A}^k \cup \mathbb{B}^{n-k}$.*

Suppose that $|\lambda| \neq 1$, so that $\lambda \neq \frac{1}{\overline{\lambda}}$. Set $\alpha \equiv \frac{1}{\overline{\lambda}}$. Then $x \in V_\lambda$ if and only if $L_k x \in V_\alpha$. Moreover, V_λ and V_α are orthogonal subspaces and $\dim(V_\lambda) = \dim(V_\alpha)$. Let $\mathcal{A} = \{x_1, ..., x_t\}$ be an orthonormal basis of V_λ. Then $(L_k x_p)^* (L_k x_q) = x_p^* x_q$ for every $p, q = 1, ..., t$. Hence, $\mathcal{B} = \{L_k x_1, ..., L_k x_t\}$ is an orthonormal basis for V_α. Moreover, $\{x_1, ..., x_t, L_k x_1, ..., L_k x_t\}$ is also an orthonormal set and is a basis of span$(\mathcal{A} \cup \mathcal{B})$. Let $\mathcal{C}$ be as in equation 1 so that span$(\mathcal{C})$ = span$(\mathcal{A} \cup \mathcal{B})$. Then $\mathcal{C}$ is also a basis of span$(\mathcal{A} \cup \mathcal{B})$. Now, each y_p is a $(k, 0)$ vector and each z_q is a $(0, n-k)$ vector. Hence, we have $t \leq k$ and $t \leq n - k$. Notice that for every $p, q = 1, ..., t$ we have (1) $y_p^* z_q = 0$, (2) if $p \neq q$, then we have $y_p^* y_q = z_p^* z_q = 0$, and (3) $y_p^* y_p = z_p^* z_p = 2$. For $p = 1, ..., t$, let $u_p = \frac{1}{\sqrt{2}} y_p$ and let $v_p = \frac{1}{\sqrt{2}} z_p$. Then $\{u_1, ..., u_t, v_1, ..., v_t\}$ is an orthonormal basis of span$(\mathcal{A} \cup \mathcal{B})$.

Lemma 7. *Let $A \in \mathcal{O}_{L_k}$ be normal. Let $\lambda_1, ..., \lambda_t$ be the distinct eigenvalues of A with $|\lambda_1| \geq \cdots \geq |\lambda_t| > 1$. For each $p = 1, ..., t$, set $\alpha_p = \frac{1}{\overline{\lambda_p}}$. Then, for each $p, q = 1, ...t$ we have (i) V_{λ_p} and V_{α_q} are orthogonal, (ii) if $p \neq q$, then V_{λ_p} and V_{λ_q} are orthogonal, and (iii) if $p \neq q$, then V_{α_p} and V_{α_q} are orthogonal. For each $p = 1, ..., m_p$, if $\mathcal{A}_p = \left\{x_{p1}, ..., x_{pm_p}\right\}$ is an orthonormal basis for V_{λ_p}, then $\mathcal{B}_p = \left\{L_k x_{p1}, ..., L_k x_{pm_p}\right\}$ is an orthonormal basis for V_{α_p}. Set $y_{pq} = \frac{1}{\sqrt{2}}(x_{pq} + L_k x_{pq})$, set $z_{pq} = \frac{1}{\sqrt{2}}(x_{pq} - L_k x_{pq})$, and set $\mathcal{C}_p = \left\{y_{p1}, ..., y_{pm_p}, z_{p1}, ..., z_{pm_p}\right\}$. Then each $y_{pq} \in \mathbb{A}^k$, each $z_{pq} \in \mathbb{B}^{n-k}$, and each $\mathcal{C}_p$ is an orthonormal basis of span$(\mathcal{A}_p \cup \mathcal{B}_p)$. Moreover, we have $\sum_{p=1}^t dim(V_{\lambda_p}) \leq k$ and $\sum_{p=1}^t dim(V_{\lambda_p}) \leq n - k$.*

For $p = 1, ..., t$, let y_p and z_p be as in equation (1), let $u_p = \frac{1}{\sqrt{2}} y_p$, and let $v_p = \frac{1}{\sqrt{2}} z_p$ so that $\{u_1, ..., u_t, v_1, ..., v_t\}$ is an orthonormal basis of span$(\mathcal{A} \cup \mathcal{B})$. Hence, $Au_p = \frac{1}{\sqrt{2}} A(x_p + L_k x_p) = \frac{1}{\sqrt{2}}\left(\lambda x_p + \frac{1}{\lambda} L_k x_p\right) = \frac{1}{\sqrt{2}}\left(\lambda \frac{1}{2}(y_p + z_p) + \frac{1}{\lambda}\frac{1}{2}(y_p - z_p)\right) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) u_p + \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) v_p$. Similarly, $Av_p = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) u_p + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) v_p$.

Theorem 8. *Let $A \in \mathcal{O}_{L_k}$ be normal. Let $\sigma(A) = \left\{\lambda_1, ..., \lambda_t, \frac{1}{\lambda_1}, ..., \frac{1}{\lambda_t}\right\}$ with $|\lambda_1| \geq \cdots \geq |\lambda_t| > 1$. For each $p = 1, ..., t$, if the multiplicity of λ_p is s_p, then the multiplicity of $\frac{1}{\lambda_p}$ is also s_p. Moreover, we have $n = 2k = 2\sum_{p=1}^{t} s_p$. For $p = 1, ..., t$, set $\alpha_p = \frac{1}{\lambda_p}$, set $\beta_p = \frac{1}{2}(\lambda_p + \alpha_p)$, set $\delta_p = \frac{1}{2}(\lambda_p - \alpha_p)$, set $D_p = \beta_p I_{s_p}$, set $E_p = \delta_p I_{s_p}$, set $D = D_1 \oplus \cdots \oplus D_t$, and set $E = E_1 \oplus \cdots \oplus E_t$. There exists a unitary $P \in \mathcal{O}_{L_k}$ such that $P^*AP = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$.*

Proof. For $p = 1, ..., t$, set $\alpha_p = \frac{1}{\lambda_p}$, set $\beta_p = \frac{1}{2}(\lambda_p + \alpha_p)$, and set $\delta_p = \frac{1}{2}(\lambda_p - \alpha_p)$. Because $\dim(V_{\lambda_p}) = \dim(V_{\alpha_p})$, we have $k = n - k$ so that $n = 2k = 2\sum_{p=1}^{t} s_p$, as desired.

Let y_{pq} and z_{pq} be as in Lemma 7. Set $Y_p = [y_{p1} \cdots y_{pm_p}]$, set $Z_q = [z_{q1} \cdots z_{qm_q}]$, and set $P = [Y_1 \cdots Y_t \ Z_1 \cdots Z_t]$. Set $D_p = \operatorname{diag}(\beta_p, ..., \beta_p) \in M_{m_p}(\mathbb{C})$, set $E_p = \operatorname{diag}(\delta_p, ..., \delta_p) \in M_{m_p}(\mathbb{C})$, set $D = D_1 \oplus \cdots \oplus D_t$, set $E = E_1 \oplus \cdots \oplus E_t$., and set $M = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$. Because each $y_{pq} \in \mathbb{A}^k$ and each $z_{pq} \in \mathbb{B}^k$, there exist unitary $Q_1 \in M_k(\mathbb{C})$ and $Q_2 \in M_{n-k}(\mathbb{C})$ such that $P = Q_1 \oplus Q_2$. Thus, P is unitary and $P \in \mathcal{O}_{L_k}$. Moreover, one checks that $AP = PM$. □

Let $A \in \mathcal{O}_{L_k}$ be normal, and that $\sigma(A) = \left\{\lambda_1, ..., \lambda_t, \frac{1}{\lambda_1}, ..., \frac{1}{\lambda_t}\right\}$ with $|\lambda_1| \geq \cdots \geq |\lambda_t| > 1$. For $p = 1, ..., t$, let λ_p, α_p, β_p, δ_p, D_p, and E_p be as in Theorem 8. Write $\lambda_p = |\lambda_p| e^{i\theta_p}$. Then $\alpha_p = \frac{1}{|\lambda_p|} e^{i\theta_p}$, $\beta_p = \frac{1}{2}\left(|\lambda_p| + \frac{1}{|\lambda_p|}\right) e^{i\theta_p}$, and $\delta_p = \frac{1}{2}\left(|\lambda_p| - \frac{1}{|\lambda_p|}\right) e^{i\theta_p}$. Set $C_p = \frac{1}{2}\left(|\lambda_p| + \frac{1}{|\lambda_p|}\right) I_{s_p}$, set $W_p = e^{i\theta_p} I_{s_p}$, and set $F_p = \frac{1}{2}\left(|\lambda_p| - \frac{1}{|\lambda_p|}\right) I_{s_p}$. Let $C = C_1 \oplus \cdots \oplus C_t$, let $W = W_1 \oplus \cdots \oplus W_t$, and let $F = F_1 \oplus \cdots \oplus F_t$. Then $D = CW$ and $E = FW$. Set $U = W \oplus W$. Then U is unitary and $U \in \mathcal{O}_{L_k}$. Moreover, $MU^* = \begin{bmatrix} C & F \\ F & C \end{bmatrix}$. Let $X = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \in M_n(\mathbb{C})$. One checks that X is unitary and that $XMU^*X^* = (C + F) \oplus (C - F)$. Now, $C + F = |\lambda_1| I_{s_1} \oplus \cdots \oplus |\lambda_t| I_{s_t}$ and $C - F = \frac{1}{|\lambda_1|} I_{s_1} \oplus \cdots \oplus \frac{1}{|\lambda_t|} I_{s_t}$, so that XMU^*X^* is positive definite. If A is positive definite, then we may take each $\theta_p = 0$, so that $W = I$.

Corollary 9. *Let $A \in \mathcal{O}_{L_k}$ be normal. Let $\sigma(A) = \left\{\lambda_1, ..., \lambda_t, \frac{1}{\lambda_1}, ..., \frac{1}{\lambda_t}\right\}$ with $|\lambda_1| \geq \cdots \geq |\lambda_t| > 1$. For $p = 1, ..., t$, suppose that λ_p has multiplicity s_p. Set $\beta_p = \frac{1}{2}\left(|\lambda_p| + \frac{1}{|\lambda_p|}\right)$, set $\delta_p = \frac{1}{2}\left(|\lambda_p| - \frac{1}{|\lambda_p|}\right)$, set $D_p = \beta_p I_{s_p}$, set $E_p = \delta_p I_{s_p}$, set $D = D_1 \oplus \cdots \oplus D_t$, and set $E = E_1 \oplus \cdots \oplus E_t$. There exist unitary $P, Q \in \mathcal{O}_{L_k}$ such that $P^*AQ = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$. If A is also positive definite, then we may take $Q = P$.*

Notice that both D and E are positive definite, and that $D^2 - E^2 = I$. Moreover, both D and E are determined by the eigenvalues of A (assuming the eigenvalues do not lie on the unit circle).

Let $A \in \mathcal{O}_{L_k}$ be normal. Suppose that the distinct eigenvalues of A that are on the unit circle are $\lambda_1, ..., \lambda_t$. Let $V_\lambda = \operatorname{span}\{V_{\lambda_1}, ..., V_{\lambda_t}\}$. Lemma 6 guarantees that for $p = 1, ..., t$, each V_{λ_i} has orthonormal basis in $\mathbb{A}^k \cup \mathbb{B}^{n-k}$. Let $\mathcal{A}_1 = \{x_1, ..., x_r\}$ be the set of such vectors in $\mathbb{A}^k$ and let $\mathcal{B}_1 = \{y_1, ..., y_s\}$ be the set of such vectors in $\mathbb{B}^{n-k}$. Then $j \equiv k - r = n - k - s$.

Extend $\mathcal{A}_1$ to a basis of $\mathbb{A}^k$, say $\mathcal{A}_2 = \{x_1, ..., x_r, ..., x_k\}$ and extend $\mathcal{B}_1$ to a basis of $\mathbb{B}^{n-k}$, say $\mathcal{B}_2 = \{z_1, ..., z_j, y_1 ..., y_s\}$. Notice that $V_\lambda^\perp = \text{span}\{x_{r+1}, ..., x_k, z_1, ..., z_j\}$. If $x \in V_\lambda$, then $Ax \in V_\lambda$. In particular, if $x \in \text{span}(\mathcal{A}_1)$, then $Ax \in \text{span}(\mathcal{A}_1)$; and if $x \in \text{span}(\mathcal{B}_1)$, then $Ax \in \text{span}(\mathcal{B}_1)$. If $x \in V_\lambda^\perp$, then $Ax \in V_\lambda^\perp$. If $x \in V_\lambda$ and if $y \in V_\lambda^\perp$, then $x^* Ay = 0 = y^* Ax$. Let $X = [x_1 \cdots x_k]$, let $Z = [z_1 \cdots z_j]$, let $Y = [y_1 \cdots y_s]$, and let $P = [X \ Z \ Y]$. There exist unitary $Q_1 \in M_k(\mathbb{C})$ and unitary $Q_2 \in M_{n-k}(\mathbb{C})$ such that $P = Q_1 \oplus Q_2$, so that $P \in \mathcal{O}_{L_k}$. For $p = 1, ..., r$, suppose that $Ax_p = \lambda_{m_p} x_p$, and for $q = 1, ..., s$, suppose that $Ay_q = \lambda_{t_q} y_q$. Let $\Sigma_1 = \text{diag}(\lambda_{m_1}, ..., \lambda_{m_r})$ and let $\Sigma_2 = \text{diag}(\lambda_{t_1}, ..., \lambda_{t_s})$. There exists $N \in M_{2j}(\mathbb{C})$ such that $P^* AP = \Sigma_1 \oplus N \oplus \Sigma_2$. Because $A \in \mathcal{O}_{L_k}$, we have $N \in \mathcal{O}_{L_j}$. Moreover, the eigenvalues of N do not lie on the unit circle (as the eigenvalues of N are the remaining eigenvalues of A). Corollary 9 guarantees that there exist unitary $P_1, Q_1 \in O_{L_j}$, positive definite diagonal $D, E \in M_j(\mathbb{C})$ such that $D^2 - E^2 = I$ and $P_1^* N Q_1 = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$. Set $P_2 = \Sigma_1 \oplus P_1 \oplus \Sigma_2$ and set $Q_2 = I_t \oplus Q_1 \oplus I_s$. Then, P_2 and Q_2 are both unitary and both are in $\mathcal{O}_{L_k}$. Moreover, $P_2^* P^* APQ_2 = I_t \oplus P_1^* N Q_1 \oplus I_s$.

Theorem 10. *Let $A \in \mathcal{O}_{L_k}$ be normal. There exist nonnegative integers r and s, unitary $P, Q \in \mathcal{O}_{L_k}$, positive definite diagonal $D, E \in M_{k-r}(\mathbb{C})$ such that (1) $k - r = n - k - s$, (2) $D^2 - E^2 = I_{k-r}$, and (3) $A = P \left(I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s \right) Q^*$. If A is also positive definite, then we may take (a) $Q = P$ and (b) take $D - E$ to be positive definite.*

Let $B \in \mathcal{O}_{L_k}$ be given. Then BB^* and B^*B are also in $\mathcal{O}_{L_k}$. Moreover, both BB^* and B^*B are positive definite, and BB^* is similar to B^*B. For a normal $A \in \mathcal{O}_{L_k}$, notice that the positive definite matrices D and E in Theorem 10 are determined by the eigenvalues of A that are not lying on the unit circle. Once these are determined, the integers r and s are easily determined. Theorem 10 guarantees that there exist unitary $P, Q \in \mathcal{O}_{L_k}$ such that $P^* BB^* P = I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s = Q^* B^* BQ$. Let $Z = QP^*$. Then $Z \in \mathcal{O}_{L_k}$ and is also unitary. Moreover, $BB^* = Z^* B^* BZ$. Set $C = BZ$. Then $C \in \mathcal{O}_{L_k}$ and $C^* C = Z^* B^* BZ = BB^* = CC^*$, so that C is normal, as well. Theorem 10 now shows the following.

Corollary 11. *Let $A \in \mathcal{O}_{L_k}$ be given. There exist nonnegative integers r and s, unitary $P, Q \in \mathcal{O}_{L_k}$, and positive definite diagonal $D, E \in M_{k-r}(\mathbb{C})$ such that $k - r = n - k - s$, $D^2 - E^2 = I_{k-r}$, and $A = P \left(I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s \right) Q^*$.*

Let $A \in \mathcal{O}_{L_k}$ be given. Let the integers r and s, the unitary $P, Q \in \mathcal{O}_{L_k}$, and positive definite diagonal matrices $D, E \in M_{k-r}(\mathbb{C})$ be as in Corollary 11. Set $X = I_r \oplus \begin{bmatrix} D & E \\ E & D \end{bmatrix} \oplus I_s$. Set $j = k - r$. Let $V = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \in M_{2j}(\mathbb{C})$. One checks that $K \equiv \begin{bmatrix} D & E \\ E & D \end{bmatrix} = V^* \begin{bmatrix} D+E & 0 \\ 0 & D-E \end{bmatrix} V$. Write $D + E = \text{diag}(a_1, ..., a_j)$ and write $D - E = \text{diag}(b_1, ..., b_j)$. Then for each $p = 1, .., j$, we have $b_p = \frac{1}{a_p}$. Let $c_p = \ln(a_p)$, so that $\ln(b_p) = -c_p$. Let $C = \text{diag}(c_1, ..., c_j)$, and let $G = C \oplus -C$. Then $K = V^* e^G V = e^{V^* GV}$. Now, $V^* GV = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$.

Corollary 12. *Let $A \in \mathcal{O}_{L_k}$ be given. There exist nonnegative integers r and s, unitary $P, Q \in \mathcal{O}_{L_k}$, positive definite diagonal $D \in M_{k-r}(\mathbb{C})$ such that $k - r = n - k - s$, $F = 0_r \oplus \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \oplus 0_s$, and $A = Pe^F Q^*$.*

Notice that in Corollary 11, $A = PXQ^* = PXP^*(PQ^*)$, that PXP^* is positive definite, that PQ^* is unitary, and that both PXP^* and PQ^* are in $\mathcal{O}_{L_k}$. This is the polar decomposition of A. We present a different proof. First, notice that if $A \in \mathcal{O}_{L_k}$, then $L_k A^* L_k = A^{-1}$. Taking the conjugate, we have $L_k \left(\overline{A}\right)^* L_k = \left(\overline{A}\right)^{-1}$ so that $\overline{A} \in \mathcal{O}_{L_k}$. Similarly, it can be shown that A^T and A^* are also in $\mathcal{O}_{L_k}$.

Theorem 13. *Let $A \in \mathcal{O}_{L_k}$ be given. There exist $P, U \in \mathcal{O}_{L_k}$ with P positive definite and U unitary such that $A = PU$.*

Proof. Let $A \in \mathcal{O}_{L_k}$ be given. Then $A^* \in \mathcal{O}_{L_k}$ and hence, $AA^* \in \mathcal{O}_{L_k}$. Notice that AA^* is positive definite. Let $q(x)$ be a polynomial that interpolates $\sqrt{x}$ in the joint spectrum of AA^* and $(AA^*)^{-1}$. Then we may choose $q(x)$ to be a real polynomial. If $q(AA^*) = P$, then P is positive definite and $q\left((AA^*)^{-1}\right) = P^{-1}$. Now, $P^{-1} = q\left((A^*A)^{-1}\right) = q(\Lambda_H(A^*A)) = \Lambda_H(q(A^*A)) = \Lambda_H(P)$ and $P \in \mathcal{O}_{L_k}$. Set $U \equiv P^{-1}A$. Then $U \in \mathcal{O}_{L_k}$ (as both A and P are), $U^*U = P^{-1}A^*AP^{-1} = I$, and $A = PU$, as desired. □

Let $H \in \mathcal{H}_n$ be given, and suppose that there exists a nonsingular $X \in M_n(\mathbb{C})$ such that $H = X^* L_k X$. Let $A \in \mathcal{O}_H$ be given. Lemma 3 guarantees that $XAX^{-1} \in \mathcal{O}_{L_k}$. Moreover, Lemma 13 ensures that $XAX^{-1} = PU$, with $P, U \in \mathcal{O}_{L_k}$, P is positive definite, and U is unitary. Now, $A = \left(X^{-1}PX\right)\left(X^{-1}UX\right)$, and notice that both $X^{-1}PX$ and $X^{-1}UX$ are in $\mathcal{O}_H$. Notice also that both $X^{-1}PX$ and $X^{-1}UX$ are diagonalizable, with $X^{-1}PX$ having positive eigenvalues and $X^{-1}UX$ having eigenvalues that are on the unit circle. Such a factorization is known as the Iwasawa decomposition [5, Lemma 3.12].

Corollary 14. *Let $H \in \mathcal{H}_n$ be given, and suppose that there exists a nonsingular $X \in M_n(\mathbb{C})$ such that $H = X^* L_k X$. Then $A \in \mathcal{O}_H$ if and only if there exist $W, V \in \mathcal{O}_H$ such that both W and V are diagonalizable, W has positive eigenvalues, V has eigenvalues that are on the unit circle, and $A = WV$.*

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