Modeling Dynamics of Political Parties with Poaching from One Party

Priscilla S. Macansantos
University of the Philippines Baguio, Baguio City 2600, Philippines

psmacansantos@up.edu.ph

Abstract. Dynamical social phenomena have been modeled using differential systems employed in mathematical models of epidemiology. In Misra (2012), an epidemiological model is used to study the "spread" of two political parties. In Nyabadzaet. al (2016), the authors point out that the Misra model may be generalized by considering switching between political parties. We take off from these to consider a model in which there are two parties, with a third party recruiting only from one, hence acting as a renegade to one of the two parties. We locate equilibria and consider local stability particularly of the co-existence/endemic equilibrium. Finally, we look at parameter values that result in the recruitment activities of the third party adversely affecting the group it recruits from.

Keywords: mathematical modeling, political party dynamics

1.Introduction

A number of dynamical social phenomena have been modeled using differential systems employed in mathematical models of epidemiology. In Misra [1], an epidemiological model is used to study the "spread" of two political parties. Using simplifying assumptions, the following system of equations is studied:

$$\frac{dV}{dt} = \mu N - \beta_1 V \frac{B}{N} - \beta_2 V \frac{C}{N} - \mu V$$
$$\frac{dB}{dt} = \beta_1 V \frac{B}{N} - \theta_1 B \frac{C}{N} + \theta_2 C \frac{B}{N} - \mu B \qquad (1)$$
$$\frac{dC}{dt} = \beta_2 V \frac{C}{N} + \theta_1 B \frac{C}{N} - \theta_2 C \frac{B}{N} - \mu C$$

In the above system, *N* is total population, *V* is the susceptible/voting class, *B* refers to the first political party "class", *C* the second party "class," β_i are the recruitment rates from *V* to *B* and *C*, respectively, while θ_i are the recruitment rates from *B* to *C* and *C* to *B*, respectively, and μ is the uniform rate of departure from each class. The model is then studied for equilibrium states, stability and persistence.

In Nyabadzaet. al [2], the authors point out that the Misra model may be refined by considering switching between political parties. In the model studied by Misra [1], the movements of members from political party B to political party C andvice versa, are considered but the net movement is instead utilized by assuming that $\theta_1 - \theta_2 = \theta$ (a constant) which implies that the movement of members is either from party *B* to party *C* or from party *C* to party *B*. Nyabadza *et al* [2] remodel these movements through switching functions to capture how individuals switch between parties. Two switching functions that depend on the size of each political party and some parameters are introduced. These functions generalize the Misra paper [1] in which the net movement was assumed to be unidirectional, or in favor of a given political party. In addition to some results obtained in Misra [1], additional information regarding how the behavior of the population size is dependent on the switching parameters is demonstrated. The inclusion of switching functions improved the Misra [1] model.

In Romero [3], the point is made that third parties are influential in shaping politics in certain countries, hence the interest in studying the spread of a third party influence in a voting population where it is assumed that party members/activists are more influential in recruiting new third party voters than non-member third party voters. The study uses an epidemiological approach to develop a theoretical model with nonlinear ordinary differential equations. The use of mathematics and mathematical models to study political party dynamics has yielded interesting observations, see for instance Laver [4], Polk and Karreth [5] and De la Poza *et al* [6]. In recent US elections, the likelihood of third party activity drastically influencing election outcome has been considered.

We will consider a scenario where there are two major political parties, with a third party recruiting members from one of these, thus taking the role of "renegade," and potentially affecting the outcome in terms of numbers ofvoters/adherents for either party. We will set up the model and examine equilibrium points, as well as characterize each in terms of stability, based on relationships between parameter values. The paper is organized as follows: In Section 2, we will set up the model for two major political parties recruiting from the pool of voters. Section 3 will discuss equilibrium values of the model, as well as conditions for the existence of these. Section 4 will study stability of the equilibrium points. In Section 5, we further investigate the stability of the coexistence equilibrium. Section 6 will give conditions on the parameter values that lead to the situation in which third party activity will adversely affect the outcome for the party it is “poaching” from. Section 7 gives the resultof two simulations demonstrating our results.

2. Model: Two major parties with a third recruiting from one of the two majors

We consider the case where there are two major/established parties and one of these has a “renegade” group forming a third party. Recruitment by the third party is from only one of the major parties. The model consists of the following equations:

$$\frac{dV}{dt} = \mu N - \beta_1 V\frac{B}{N} - \beta_2 V\frac{C}{N} - \mu V$$
$$\frac{dB}{dt} = \beta_1 V\frac{B}{N} - \theta B\frac{R}{N} - \mu B$$
$$\frac{dC}{dt} = \beta_2 V\frac{C}{N} - \mu C, \quad \frac{dR}{dt} = \theta B\frac{R}{N} - \mu R \quad (2)$$

With N=V+B+C+R, adding the three equations yields $\frac{dN}{dt}$=0, that is, it is assumed that N is constant. To simplify, we let v=V/N, b=B/N, c=C/N, r=R/N, and we get the following system

$$\frac{dv}{dt} = \mu - \beta_1 vb - \beta_2 vc - \mu v$$
$$\frac{db}{dt} = \beta_1 vb - \theta br - \mu b \quad (3)$$
$$\frac{dc}{dt} = \beta_2 vc - \mu c, \qquad \frac{dr}{dt} = \theta br - \mu r$$

Note that since $v+b+c+r=1$, then $v=1-b-c-r$. Furthermore, $\mu, \beta_1, \beta_2, \theta \geq 0$. With the assumption that N is constant, the equivalent system is

$$\frac{db}{dt} = \beta_1(1-b-c-r)b - \theta br - \mu b \quad (4)$$
$$\frac{dc}{dt} = \beta_2(1-b-c-r)\,c - \mu c, \qquad \frac{dr}{dt} = \theta br - \mu r$$

It is naturally assumed that $b(0) \geq 0$, $c(0) \geq 0$, $r(0) \geq 0$, $0 \leq b(t) \leq 1$, $0 \leq c(t) \leq 1$, $0 \leq r(t) \leq 1$.

3.Equilibrium Points

The following are equilibrium points for the system:

i. E_0=(0,0,0)

ii. E_1= (0,1-μ/β_2,0), this exists for $\beta_2 > \mu$

iii. E_2= (1- μ/β_1,0, 0), this exists for $\beta_1 > \mu$

iv. $E_4 = (b^*, c^*, r^*)$, where $b^* = \frac{\mu}{\theta}, c^* = 1 - \frac{\beta_1}{\beta_2} b^* - \frac{\mu}{\beta_2}, r^* = b^{*}(\frac{\beta_1}{\beta_2} - 1)$

There is a set of equilibria corresponding to $r = 0$ with $\bar{E} = (\bar{b}, \bar{c}, 0)$ such that $\bar{b} + \bar{c} = 1 - \frac{\mu}{\beta_2}$.

We observe that in order for the co-existence point to be realized, we must have $b^* = \frac{\mu}{\theta}$, so necessarily $\theta > \mu$;$r^* = b^{*}(\frac{\beta_1}{\beta_2} - 1)$ hence$\beta_1 > \beta_2$; $c^* = 1 - \frac{\beta_1}{\beta_2} b^* - \frac{\mu}{\beta_2}$, hence $\frac{\beta_2}{\beta_1 + \theta} > \frac{\mu}{\theta} = b^*$.

4. Stability of Equilibria

To investigate stability of the equilibrium points, the Jacobian for the system is

$$J(b,c,r) = \begin{bmatrix} \beta_1(1 - 2b - c - r) - \theta r - \mu & -\beta_1 b & -\beta_1 b - \theta b \\ -\beta_2 c & \beta_2(1 - 2c - b - r) - \mu & 0 \\ \theta r & 0 & \theta b - \mu \end{bmatrix}$$

For the stability of $E_0 = (0,0,0)$, note that $J(0,0,0) = \begin{bmatrix} \beta_1 - \mu & 0 & 0 \\ 0 & \beta_2 - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$, so with eigenvalues $\beta_1 - \mu. \beta_2 - \mu$, $-\mu$,note that

a. If E_2= (1- μ/β_1,0, 0) exists, then $\beta_1 > \mu$, hence if E_2 exists, $E_0 = (0,0,0)$ is unstable
b. If E_1= (0,1- μ/β_2, 0) exists, then $\beta_2 > \mu$, hence if E_1 exists, $E_0 = (0,0,0)$ is unstable
c. Hence, if at least one of E_1 *or* E_2exists, then $E_0 = (0,0,0)$ is unstable

For stability of E_1 = (0,1- μ/β_2 ,0), we have $J(E_1) = \begin{bmatrix} \beta_1 \frac{\mu}{\beta_2} - \mu & 0 & 0 \\ -\beta_2 + \mu & -\beta_2 + \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$.

Eigenvalues are $\beta_1 \frac{\mu}{\beta_2} - \mu, -\beta_2 + \mu, -\mu$. Note that if E_1 exists, then $\beta_2 > \mu$, hence if E_1 exists, then the last two eigenvalues would be negative. For the first eigenvalue to be negative, $\beta_2 > \beta_1$ and stability will follow.But for the endemic/coexistence equilibrium to exist, necessarily, $\beta_2 < \beta_1$. Thus, if coexistence occurs, E_1= (0,1- μ/β_2, 0) is unstable.

For stability of E_2, we note that $J(E_2) = \begin{bmatrix} -\beta_1 + \mu & -\beta_1 + \mu & -\beta_1 + \mu - \theta + \frac{\mu}{\beta_1} \\ 0 & \mu \frac{\beta_2}{\beta_1} - \mu & 0 \\ 0 & 0 & \theta(1 - \frac{\mu}{\beta_1}) - \mu \end{bmatrix}$. The eigenvalues are$-\beta_1 + \mu, \mu \frac{\beta_2}{\beta_1} - \mu, \theta\left(1 - \frac{\mu}{\beta_1}\right) - \mu$. If the endemic equilibrium exists, $\beta_2 < \beta_1$, hence assuming the coexistence/ endemic equilibrium occurs, stability for E_2will require, referring to the third eigenvalue, $\theta\left(1 - \frac{\mu}{\beta_1}\right) - \mu < 0$, that is, $1 - \frac{\mu}{\beta_1} < \frac{\mu}{\theta}$.

As for equilibrium points $\overline{E} = \{(\bar{b}, \bar{c}, 0)$ such that $\bar{b} + \bar{c} = 1 - \frac{\mu}{\beta_2}\}$, we have for each member of this collection, $J(\bar{E}) = \begin{bmatrix} -\beta_1\bar{b} & -\beta_1\bar{b} & -\beta_1\bar{b} - \theta\bar{b} \\ -\beta_2\bar{c} & -\beta_2\bar{c} & 0 \\ 0 & 0 & \theta\bar{b} - \mu \end{bmatrix}$. For equilibria $\overline{E}$ to exist, it is necessary that $\beta_1 = \beta_2$. With this observation, we note that $Det\ (\lambda I - J(\bar{E})) =$ $\det\begin{bmatrix} \lambda + \beta_1\bar{b} & \beta_1\bar{b} & \beta_1\bar{b} + \theta\bar{b} \\ \beta_2\bar{c} & \lambda + \beta_2\bar{c} & 0 \\ 0 & 0 & \lambda - \theta\bar{b} + \mu \end{bmatrix}$. The eigenvalues are $\theta\bar{b} - \mu$, $\mu - \beta_1$ and 0. If $\bar{b} > \mu/\theta$, then the equilibria are unstable. A change in stability occurs at $\bar{b} = \frac{\mu}{\theta}$. which is also the value of b^* for the coexistence equilibrium.

5.Stability of Coexistence Equilibrium

The coexistence equilibrium is $E_4 = \left(\frac{\mu}{\theta}, 1 - \frac{\beta_1}{\beta_2}b^* - \frac{\mu}{\beta_2}, b^*(\frac{\beta_1}{\beta_2} - 1)\right)$, with $\theta > \mu$, $\beta_1 > \beta_2$, $\frac{\beta_2}{\beta_1+\theta} > \frac{\mu}{\theta} = b^*$. For more compact notation, let $\delta = \frac{\beta_1}{\beta_2}$, $\omega = \frac{\mu}{\beta_2}$, $\kappa = \beta_1 b^*$.
Proceeding with the Jacobian,

$$J(E_4) = \begin{bmatrix} \beta_1(1 - 2b^* - c^* - r^*) - \theta r^* - \mu & -\beta_1 b^* & -\beta_1 b^* - \theta b^* \\ -\beta_2 c^* & \beta_2(1 - 2c^* - b^* - r^*) - \mu & 0 \\ \theta r^* & 0 & \theta b^* - \mu \end{bmatrix}$$

$$= \begin{bmatrix} -\beta_1 b^* & -\beta_1 b^* & -\beta_1 b^* - \mu \\ -\beta_2 + \beta_1 b^* + \mu & -\beta_2 + \beta_1 b^* + \mu & 0 \\ \mu(\delta - 1) & 0 & 0 \end{bmatrix}$$

Now, $\lambda I - J(E_4) = \begin{bmatrix} \lambda + \kappa & \kappa & \kappa + \mu \\ \beta_2 - \kappa - \mu & \lambda + \beta_2 - \kappa - \mu & 0 \\ -\mu(\delta - 1) & 0 & \lambda \end{bmatrix}$, and so $\mathrm{Det}(\lambda I - J(E_4))$ is

$$\lambda[(\lambda + \kappa)(\lambda + \beta_2 - \kappa - \mu) - \kappa(\beta_2 - \kappa - \mu)] + (\lambda + \beta_2 - \kappa - \mu)(\kappa + \mu)\,\mu(\delta - 1))$$
$$= \lambda^3 + \lambda^2(\beta_2 - \mu) + \lambda\,\kappa(\beta_2 - \kappa - \mu) - \lambda\,\kappa(\beta_2 - \kappa - \mu) + \lambda\,(\kappa + \mu)\mu(\delta - 1) +$$
$$+ (\kappa + \mu)(\beta_2 - \kappa - \mu)\,\mu(\delta - 1)$$

Recall the Routh-Hurwitz Criterion for stability, namely:
For $\mathrm{Det}(\lambda I - J) = \lambda^3 + a_1\lambda^2 + a_2\,\lambda + a_3$, sufficient conditions for stability are $a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$. We have $a_1 = \beta_2 - \mu > 0$, which is necessary for the equilibrium E_4 to exist. Also, $a_2 = (\kappa + \mu)\mu(\delta - 1)$, $a_1 a_2 = (\beta_2 - \mu)(\kappa + \mu)\mu(\delta - 1) > (\kappa + \mu)(\beta_2 - \kappa - \mu)\,\mu(\delta - 1) = a_3$.
Note that $a_3 = (\kappa + \mu)(\beta_2 - \kappa - \mu)\,\mu(\delta - 1) > 0$ provided that $\beta_2 - \kappa - \mu > 0$, that is, $\beta_2 > \kappa + \mu$, or equivalently, $\beta_2 > \mu\left(1 + \frac{\beta_1}{\theta}\right)$. Hence, for the condition $a_3 > 0$ to be true, we must have $\frac{\beta_2}{\beta_1+\theta} > \frac{\mu}{\theta}$.

We see that stable coexistence occurs provided $\beta_1 > \beta_2 > \mu$, $\theta > \mu$, and $\frac{\beta_2}{\beta_1+\theta} > \frac{\mu}{\theta}$.
This means that the recruitment rate to party *B*(the party against which poaching is being done) is greater than the recruitment rate to party *C*, the other major party, and both recruitment rates are larger than the uniform departure rate. Further, the switching rate θ from party *B* to *C* (the "renegade") is larger than the departure rate from all classes. Finally, the ratio $\frac{\beta_2}{\beta_1+\theta}$exceeds $\frac{\mu}{\theta}$, which happens to be the value of *b* at the coexistence equilibrium.

6. Conditions for the "poaching" to adversely affect the source party

We finally investigate conditions on the parameters that will result in the poaching activity yielding adverse effects on the equilibrium value of the source party, that is, $c^* > b^*$. Here, $c^* > b^*$ means

$$1 - \frac{\beta_1}{\beta_2}\frac{\mu}{\theta} - \frac{\mu}{\beta_2} > \frac{\mu}{\theta}.$$

This is equivalent to $\frac{\beta_2\theta - \beta_1\mu - \mu\theta}{\beta_2\theta} > \frac{\mu}{\theta}$, yielding $\beta_2(\theta - \mu) > \mu(\beta_1 + \theta)$ and finally, $\frac{\beta_2}{\beta_1+\theta} > \frac{\mu}{\theta-\mu}$.

Recall that condition for stability of endemic equilibrium is $\frac{\beta_2}{\beta_1+\theta} > \frac{\mu}{\theta}$, hence the stricter condition is required in order for the "third/ renegade party" to adversely affect the group it is recruiting from.

7. A Simulation

The following are trajectories corresponding to the case where $\beta_1 = 0.5$, $\beta_2 = 0.45$, $\theta = 0.15$, $\mu = 0.05$. In the case given by figures 1 and 2, we note that the "poaching" from party *B* results in party *C* having a larger share of voters at equilibrium. A different case arises when there is an increase (change) on the value of μ from 0.05 to 0.1, while the other parameters are the same. Figures 3 and 4 show that Party C does not overtake Party B despite poaching by Party R from Party B.

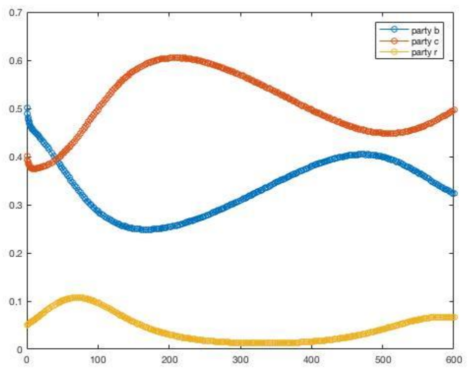

Figure 1. Switching from Party B to third Party R at a rate $\theta = 0.15$, $\mu = 0.05$ results in Party C gaining more adherents at equilibrium

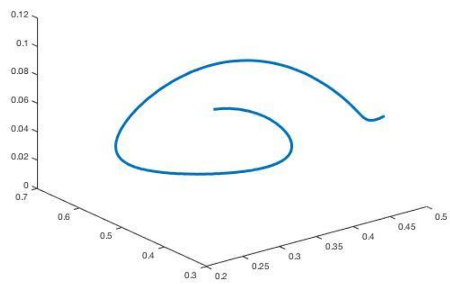

Figure 2. Three-dimensional trajectory for parties (B,C,R) with parameters $\beta_1 = 0.5$, $\beta_2 = 0.45$, $\theta = 0.15$, $\mu = 0.05$

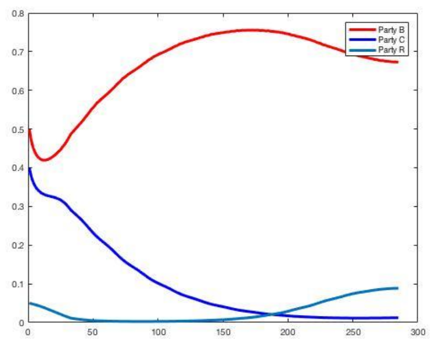

Figure 3. Switching from Party B to third Party R at a rate $\theta = 0.15$, $\mu = 0.1$, results in Party C still unable to get more adherents at

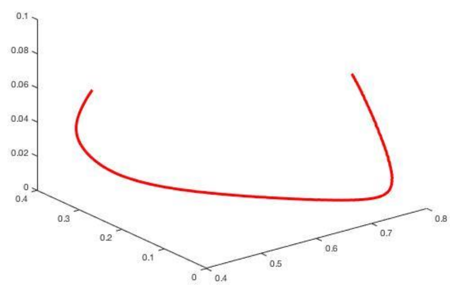

Figure 4. Three-dimensional trajectory for (B,C,R) with parameters $\beta_1 = 0.5$, $\beta_2 = 0.45$, $\theta = 0.15$, $\mu = 0.1$

equilibrium

8.Conclusion

In this study, a model of two political parties is proposed, with a third party recruiting from one of the two parties. Stable coexistence of the three parties is verified, under conditions on parameters representing recruitment rate, switching rate, and natural departure rate. Third party recruitment becomes effective against one party under stricter conditions on the parameters. Finally, it is observed that the ratio of $\frac{\mu}{\theta}$is significant as it affects stability of the coexistence equilibrium as well as that of the equilibria with no third party members.

Acknowledgments

The author acknowledges the support of the University of the Philippines for release time during the period of research and manuscript preparation.

References

[1] Misra A K 2012, A simple mathematical model for the spread of two political parties, *Nonlinear Analysis: Modelling and Control*, **17 (3)**:343–354

[2] Nyabadza F, Alassey T and Muchatibaya G 2016, Modelling the dynamics of two political parties in the presence of switching, *Springer Plus***(5):**1018

[3] Romero DM, Kribs-Zaleta C, Mubayi A and Orbe C 2011, An epidemiological approach to the spread of political third parties, *Discrete and Continuous Dynamical Systems Series B***15 (3)**1-32

[4] Laver M 2005, Policy and the dynamics of political competition, *American Political Science Review*, **99 (2):** 263-281.

[5] Polk J and Karreth J 2017, Explaining voter responses to mainstream parties' moderation strategies, *Centre for European Research (CERGU) Working Paper Series* **2017:4,** 1-43.

[6] De la Poza E, Jodar I and Pricop A 2013, Mathematical modeling of the propagation of democratic support of extreme ideologies in Spain: causes, effects, and recommendations for its stop, *Abstract and Applied Analysis* (Hindawi), **2013 (729814)** 1-8.